Investigation of lossless transmission lines by time-domain analysis

D. Mutambo, D. Mukherjee, E. Rrustamaj
Physical Electronics Group, School of Electrical, Electronic and Information Engineering, South Bank University, UK

Abstract

A direct time-domain method is presented for calculating voltage waves at terminals and transmission line terminating in an arbitrary non-linear load. By utilising a time stepping of the load voltage with a sufficiently small time interval, one may linearize the non-linear load in that interval and obtain the convergent voltage responses. Numerical examples show that this is efficient in the computation without the use of the inverse Fourier transform or the Newton-Raphson method for the lossy transmission line. This work forms a general basis for the investigation of non-uniform microstrip line such as exponential tapered stripline and thus has significant application potential in high-speed communication circuits.

1 Introduction

For high-speed digital integrated circuits, the interconnections of semiconductor devices behave as transmission lines terminated with non-linear loads. Signal delays and rising time along or at the terminals of these lines are investigated either by the direct time-domain approach [1], [2] or by the transform of the frequency-domain data into the time domain [2]-[6].

Liu and Tesehe [2] analysed the problem of a linear antenna with non-linear load by representing the induced current along the antenna by the integral of the space
and time domain, which is solved numerically with the non-linear boundary condition via the Newton-Raphson method. They also showed that a similar integral equation is obtained from the convolution of the load voltage and the inverse Laplace transform of the frequency-domain data, that is, the driving point admittance looking toward the antenna at the load terminal. A numerical scheme or the time-marching procedure yields the current of the non-linear load [2]. Generalised extensions of this formulation via the inverse Laplace transform of either the frequency-domain admittance data to quasi-matched multiconductor transmission lines [3] or the two terminal frequency-domain scattering parameters [4] are applied to the lossy, dispersive, and coupled transmission lines loaded with non-linear terminations. Standing waves in the transmission lines may be expanded by a Chebyshev polynomial series, and its expansion coefficients may be obtained numerically by solving the time dependent transmission-line equation with the non-linear load via the Newton-Raphson method [5]. Direct time-domain forward and reflected voltage waves are obtained for a lossless transmission line terminating in a time-varying resistance without the transform of the frequency-domain data and its convolution [1].

By employing a successive time-stepping algorithm that linearizes the non-linear load voltage in incremental time-stepping period, one may obtain responses of the non-linear load problems and extend them to more general cases of lossy transmission lines terminating in an arbitrary non-linear load. A numerical example of the non-linear resistor and capacitor in parallel terminated in the lossy and dispersive transmission line shows that this algorithm is efficient in computation time and gives convergent values even when the voltage-current characteristics of the non-linear loads are multivalued, that is, not monotonically increasing. The Newton-Raphson method may give more than one solution for a non-monotonically increasing non-linear load [2], [6].

2 Time Stepping of the Non-linear Load in the Lossless Transmission Line (LTL)

A lossless transmission line is connected to a transient source at \( z = 0 \) and terminated with a non-linear load at \( z = l \), as shown in Figure 1. By representing the forward and backward travelling voltage waves by \( v^+(z, t) \) and \( v^-(z, t) \) at any point \( z \) and time \( t \) on the transmission line, respectively, the total voltage and current, \( v(z, t) \) and \( i(z, t) \) are given as

\[
\begin{align*}
  v(z, t) &= v^+(z, t) + v^-(z, t) \\
  Z_c i(z, t) &= v^+(z, t) - v^-(z, t),
\end{align*}
\]

where \( Z_c \) is the characteristic impedance of the line.

Let the source at \( z = 0 \) be \( v_s(t) \), having internal resistance \( R_s \), and the load at \( z = l \) be a non-linear resistor of its conductance \( G_L(v, i) \) and the non-linear capacitor \( C_L(v, i) \) in parallel, as in Figure 1(a). Their characteristics are given in Figure 1(b).
Figure 1: (a) Transmission line terminated by a nonlinear conductor and a capacitor in parallel. (b) Nonlinear characteristics of the terminal resistor and capacitor

One may linearize the non-linear load by taking the successive time-stepping intervals, $\delta t$, in which the current-voltage relation is linearized as

$$i(l, t_q) = G_q v(l, t_q) + C_q \left. \frac{d}{dt} v(l, t) \right|_{t=t_q},$$

where $G_q$ and $C_q$ are linearized conductance and capacitance, respectively, at the $qth$ time interval, $t = t_q$. From the given non-linear characteristics in Figure 1(b), one obtains

$$G_q = \frac{i(v_q)}{v_q},$$

$$C_q = \left. \frac{dQ(v)}{dv} \right|_{v=v_q},$$

where $i(v_q)$ and $Q(v)$ are the current and the charge, respectively, at time $t_q$ and $v = v_q$. 
Substituting $i$ and $v$ from Eq. (1) into Eq. (2) and the time derivative with respect to the incremental time interval $\delta t$ as

$$\frac{dv(l,t)}{dt} \bigg|_{t_q} = \frac{v(l,t_q + \delta t) - v(l,t_q)}{\delta t},$$

one obtains the backward travelling wave as

$$v^-(l,t_q + \delta t) = \Gamma_q \bigg|_{z=l} v^+(l - c_0 \delta t, t_q)$$

$$+ \frac{Z_q}{Z_q + \delta t/C_q} \left[ v^+(l, t_q) + v^-(l, t_q) \right],$$

where $c_0$ is the propagation velocity of the waves along the transmission line,

$$v^+(l, t_q + \delta t) = v^+(l - c_0 \delta t, t_q),$$

$$\Gamma_q \bigg|_{z=l} = \frac{Z_{Lq} - Z_c}{Z_{Lq} + Z_c},$$

$$Z_{Lq} = \frac{1}{G_q + C_q/\delta t},$$

$$Z_q = \frac{1}{G_q + 1/Z_c}.$$

Equation (5) yields the voltage wave at $z = l$ and the time $t = t_q + \delta t$ reflected from the non-linear load. The first term on the right-hand side is due to the reflection of the incident wave, and the second term is due to the voltage across the non-linear capacitor. This is distributed among three parts: the equivalent resistance for the capacitor, $\delta t/C_q$, the non-linear conductance, $G_q$, and the transmission time of its resistance $Z_c$. The reflection coefficient $\Gamma_q$ in Eqs. (6a) and (6b) shows that the linearized load resistance $Z_{Lq}$, is the total resistance of two parallel resistors $1/G_q$ and $\delta t/C_q$. One may check this from the circuit equation for a capacitor expanded in the incremental time interval $\delta t$,

$$i(t_q + \delta t) = C_q \frac{v(t_q + \delta t) - v(t_q)}{\delta t}$$

or
\[ v(t_q + \delta t) = \left( \frac{\delta t}{C_q} \right) i(t_q + \delta t) + v(t_q), \]  

(8)

which shows that \( v(t_q) \) may be taken as the source voltage across the capacitor at \( t_q \) and \( \delta t / C_q \) is its equivalent resistance in the time interval of \( \delta t \).

When the transient source \( e_s(t) \) with its internal source resistance \( R_s \) is connected at \( z = 0 \), the forward voltage wave at \((z, t_q)\) is obtained by

\[ v^+(z, t_q) = v^+ \left( 0, t_q - \frac{z}{c_0} \right) = \frac{Z_c}{R_s + Z_c} e_s \left( t_q - \frac{z}{c_0} \right), \quad 0 < t_q < \frac{2l}{c_0}, \]  

(9)

where \( l \) is the total length or the line and the argument of \( e_s \) is now delayed by \( z/c_0 \).

At time \( t = t_q + \delta t \), one may represent two waves in terms of those at \( t = t_q \) as

\[ v^+(z, t_q + \delta t) = v^+(z - c_0 \delta t, t_q), \quad z > c_0 \delta t, \]  

(10a)

\[ v^-(z, t_q + \delta t) = v^-(z + c_0 \delta t, t_q), \quad z < l - c_0 \delta t. \]  

(10b)

For a sufficiently large \( t_q \), reflections from \( z = 0 \) and \( z = l \) occur and one may add up the terms due to reflections. At the source boundary of \( z = 0 \), the reflection may be accounted by the reflection coefficient,

\[ \Gamma_s \bigg| _{z=0} = \frac{v^+(0, t_q)}{v^-(0, t_q - \delta t)} = \frac{R_s - Z_c}{R_s + Z_c}, \]  

(11)

which does not depend on either \( t_q \) nor \( \delta t \). By adding the source contribution and the reflected waves, one obtains

\[ v^+(z, t_q + \delta t) = \frac{Z_c}{R_c + Z_c} e_s \left( t_q + \delta t - \frac{z}{c_0} \right) + \Gamma_s \bigg| _{z=0} v^-(c_0 \delta t - z, t_q), \quad z < c_0 \delta t, \]  

(12a)

\[ v^-(z, t_q + \delta t) = \Gamma_s \bigg| _{z=l} v^+ \left( l, t_q + \delta t - \frac{l-z}{c_0} \right) + \frac{Z_q}{Z_q + \delta t / C_q} v(l, t_q), \quad z > l - c_0 \delta t \]  

(12b)

where \( v^+(l, t_q + \delta t - (l-z) / c_0) \) may be rewritten from the relation in Eq. (10a) as \( v^+ \left( 2l - z - c_0 \delta t, t_q \right) \). \( v(l, t_q) \) is the total voltage at \( z = l \) and \( t = t_q \).

Equation (12b) reduces to Eq. (5) for \( z = l \).
For a lossy dispersive transmission line, one may use the same approach for the source and load boundaries but the characteristic impedance and the propagation velocity of the transmission line depend upon the source frequencies or, equivalently, 1/\(\Delta t\). This may be shown from the lossy transmission line equation in the time domain,

\[
\begin{align*}
\frac{\partial}{\partial z} v(z,t) + L \frac{\partial}{\partial t} i(z,t) + R_i(z,t) &= 0, \\
\frac{\partial}{\partial z} i(z,t) + C \frac{\partial}{\partial v} v(z,t) + G v(z,t) &= 0,
\end{align*}
\]

where \(R, G, L,\) and \(C\) are the serial resistance, the parallel conductance, the serial inductance, and the parallel capacitance of the lossy transmission line per unit length. Substituting Eq. (1) into Eq. (13) and the time derivative by the incremental time interval at as in Eq. (4), one obtains

\[
\frac{d}{dz} v^\pm(z, t_{q+1}) \pm j\gamma v^\pm(z, t_{q+1}) = \frac{1}{\Delta t} \frac{L}{CZ_c} \left[ v(z, t_q) \pm \frac{L}{CZ_c} i(z, t_q) \right],
\]

where \(\gamma\) and \(Z_c\) are, respectively, the propagation constant and the characteristic impedance of the lossy line that may be derived from Eq. (14) in terms of \(\Delta t\) as

\[
Z_c = \sqrt{\frac{L}{C} + \frac{R}{\Delta t}},
\]

\[
\gamma = \sqrt{\left(\frac{L}{\Delta t} + R\right) \left(\frac{C}{\Delta t} + G\right)}.
\]

In the process of time stepping \(I(z, t_q)\) and \(v(z, t_q)\) are obtained in the previous time interval \(t_q\), and \(v^\pm(z, t_{q+1})\) are to be obtained as

\[
v^\pm(z, t_{q+1}) = v^\pm(z, t_{q+1}) e^{\pm\gamma(z-z')}
\]

\[
\pm \int_{z'}^z \frac{1}{\Delta t} e^{\pm\gamma(z-u)} \left[ CZ_c v(z, t_q) \pm Li(u, t_q) \right] du
\]

where \(v^\pm(z', t_{q+1})\) are the arbitrary constants of the differential equations and are obtained from two boundary conditions at \(z = 0\) and \(z = l\), that is, Eq. (12a) with \(z = 0\) and (12b) with \(z = l\), respectively.
3 Numerical Examples

One may calculate the voltages and currents at the terminals as well as along the line for the LTL terminated with any non-linear loads via Eqs. (3), (5), (10), and (12). For the non-linear load an example of a semiconductor $P-N$ junction that may be represented by the two parallel non-linear resistors and capacitors is taken. Their characteristics are calculated and given in Figure 1(b).

The serial inductance and the parallel capacitance of the transmission line per unit length are chosen to be 1.0 $\mu H/m$, and 100 pF/m, respectively, and its length $l = 0.1$ m. The source voltage is the pulse with its rising and falling time, 0.1 ns and its pulse width 1 ns, as shown in Figure 2(a). Its internal resistance $R_s$ is chosen to be 50 $\Omega$.

Figure 2: Calculated terminal voltages of the transmission line given in Figure 1. (a) Source voltage versus time. (b) Convergence of the transient response of the terminal voltage.
Calculated load voltages are shown in Figure 2(b). As the time-stepping interval $\delta t$ decreases from 0.12 to 0.008 ns, the successive time-stepping algorithm converges to that of the Runge-Kutta method. Figure 3(b) represents the calculated load voltage when the non-linear resistor has nonmonotonic variation, as shown in Figure 3(a). Calculation for a lossy transmission line via Eq. (17) is given in Figure 4.

Because this algorithm calculates voltage waves directly in the time domain with an additional space integral without the inverse Fourier (or Laplace) transform, calculation efficiencies are greatly improved. For the convergence of this method, the time-stepping interval $\delta t$ is chosen to be sufficiently small compared with the source transient for the lossless line and smaller that $L/R$ and $C/G$, in addition, for the lossy transmission line.

The convergence properties and CPU time are shown to be dependent upon the rising time of the source and the time interval $\delta t$, which is shown in Table 1. This direct time-stepping algorithm is compared with those methods of Djordjevic, Sarkar, and Harrington [3] and Palusinski and Lee [5] in the CPU time of the HP UNIX computer in Table 1.
Figure 3: Calculated terminal voltages when the non-linear capacitor of Figure 1(b) and nonmonotonic Non-linear resistors of Figure 3(a) are terminated in parallel at the transmission line of Figure 1(a) with the same source in figure 2(a). (a) Nonmonotonic variation of Non-linear resistors. (b) Transient responses of the terminal voltages

Figure 4: Calculated terminal voltages when the serial resistance is added to the transmission line in Figure 1(a) with the same nonlinear loads and source as shown in Figure 1(b) and 2(a)
Table 1. Convergence, RMS Error, and CPU Time depend on the Rising Time and Time-Stepping Interval $\delta t$ for the LTL in Figure 1(a) Terminating in the Non-linear Loads in Figure 1(b). RMS Error is calculated by comparing these Results with those of the Rung-Kutta Method.

<table>
<thead>
<tr>
<th>Rising time (ns)</th>
<th>Time Stepping</th>
<th>RMS error (%)</th>
<th>CPU time (sec)</th>
<th>Djordjevic</th>
<th>Palusinski</th>
</tr>
</thead>
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<tr>
<td>$\delta t$ (ns)</td>
<td>0.1 0.1 0.1 0.2 0.05</td>
<td>0.12 0.05 0.008 0.017 0.005</td>
<td>5.0 1.0 1.0 1.0 1.0</td>
<td>65 35 121</td>
<td>84 47 157</td>
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<tr>
<td>TimeStepping</td>
<td>Djordjevic</td>
<td>Palusinski</td>
<td></td>
<td></td>
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<tr>
<td>0.1 0.2 0.05</td>
<td>0.017 0.035 0.009</td>
<td>0.015 0.032 0.008</td>
<td>1.0 1.0 1.0</td>
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<td></td>
</tr>
</tbody>
</table>

4 Conclusion

A direct time-domain calculation algorithm is suggested for the terminal as well as the transmission line voltages and currents when the line is terminated with arbitrary non-linear load. This load may be non-linear resistors, capacitors, and inductors, or their combinations. By utilising the time stepping of the load voltage with sufficiently small time interval, one may linearize the load in that time interval and obtain the voltage responses for the lossless and the lossy transmission line without the inverse Fourier transform. This improves the calculation efficiency. This method gives the convergent result even for a load that is not monotonically increasing, whereas the Newton-Raphson method employed by others may give more than one solution [2].

References


