On monolithic approaches to fluid-structure interactions

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Abstract

We present several aspects of a general monolithic formulation for the steady-state interaction of a viscous incompressible flow with an elastic structure undergoing large displacements. The problem is solved in a direct fully-coupled manner by a Newton–Raphson adaptive finite element method. A pseudo-solid formulation is used to manage the deformations of the fluid domain. The formulation uses fluid velocity, pressure, and pseudo-solid displacements as unknowns in the flow domain and displacements in the structural components. The adaptive formulation is verified on a problem with a closed form solution. Its capabilities are then demonstrated on different fluid-structure configurations ranging from aeronautical to biomedical fields.

Keywords: monolithic, finite-elements, adaptivity, pseudo-solid, fluid, structure.

1 Introduction

In many instances, interaction between fluids and solids is achieved through weak or loose coupling of specialized softwares. This is very cost-effective because it requires little changes to analysis modules and takes advantage of expertise accumulated in each discipline. Hydrodynamic loads obtained by CFD are transferred to the structure model to predict solid displacements which are then transferred back to the fluid module to reflect changes in the geometry. This process is repeated until convergence. However weak coupling does not always converge to a solution. For example, if the fluid added mass is much larger than that of the structure, a weak coupling treatment will result in failure of the solution procedure.

In decoupled approaches equilibrium of interface forces is approximately satisfied. This error may be amplified in sensitivity analysis or gradient-based optimization. To avoid such difficulties we consider monolithic formulations of...
the interaction between incompressible flows and structures undergoing large displacements. We discuss the following issues:

- full monolithic coupling, algorithm stability and computational savings,
- Eulerian–Lagrangian coupling via a pseudo-solid approach achieving quadratic convergence of Newton’s method,
- adaptive remeshing issues.

The proposed monolithic approach avoids approximations resulting from neglected terms in decoupled approaches.

2 Governing equations

The steady flow of an incompressible Newtonian fluid is described by the continuity, momentum equations and constitutive Newtonian law [1].

\[ \nabla \cdot \mathbf{u}_f = 0 \quad (1) \]
\[ \rho_f \mathbf{u}_f \cdot \nabla \mathbf{u}_f = \nabla \cdot \mathbf{\sigma}_f + \mathbf{f}_f \quad (2) \]
\[ \mathbf{\sigma}_f = \mu_f [\nabla \mathbf{u}_f + (\nabla \mathbf{u}_f)^T] - p \mathbf{I} \quad (3) \]

with \( \rho_f \) the fluid density, \( \mathbf{u}_f \) the fluid velocity, \( \mathbf{\sigma}_f \) the total fluid stress tensor (\( \mu_f \) being the dynamic viscosity and \( p \) the fluid pressure) and \( \mathbf{f}_f \) a body force. Eq. (1) and (2) are expressed in an Eulerian frame of reference. The flow equations are closed with the following boundary conditions,

\[ \mathbf{\sigma}_f \cdot \mathbf{n} = \mathbf{t}_f \quad \text{on } \Gamma_{f_N} \quad (4) \]
\[ \mathbf{u}_f = \mathbf{u}_f \quad \text{on } \Gamma_{f_D} \quad (5) \]

where \( \Gamma_{f_N} \) denotes a boundary with Neumann conditions (prescribed traction \( \mathbf{t}_f \)), and \( \Gamma_{f_D} \) corresponds to a Dirichlet boundary (imposed \( \mathbf{u}_f \)).

For the solid, structural equilibrium is expressed on the undeformed configuration using a total Lagrangian formulation:

\[ \nabla \cdot \mathbf{\sigma}_L + \mathbf{f} = 0 \quad (6) \]

The solid is assumed hyperelastic (St. Venant-Kirchhoff material), so that

\[ \mathbf{\sigma}_L = \mathbf{F}[\lambda_s tr(\mathbf{E}) \mathbf{I} + 2\mu_s \mathbf{E}] \quad (7) \]

with \( \mathbf{F} = \mathbf{I} + \mathbf{h} \) the deformation gradient tensor, \( \mathbf{h} \) being the displacement gradient tensor, \( \lambda_s \) and \( \mu_s \) the Lamé constants, and \( \mathbf{E} \) the full Green-Lagrange strain tensor.

\[ \mathbf{E} = \frac{1}{2}(\mathbf{h} + \mathbf{h}^T + \mathbf{h} \cdot \mathbf{h}^T) \quad (8) \]

The governing equations for the structure are supplemented by the following boundary conditions,

\[ \mathbf{\sigma}_L \cdot \mathbf{n} = \mathbf{t}_s \quad \text{on } \Gamma_{s_N} \quad (9) \]
\[ \chi_s = \chi_s \quad \text{on } \Gamma_{s_D} \]

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where $\Gamma_N^s$ denotes a Neumann boundary (prescribed traction $\bar{t}_s$), and $\Gamma_D^s$ denotes a Dirichlet boundary (imposed $\bar{u}_s$).

To manage the deformation of the fluid domain, we adapt the pseudo-solid approach of Sackinger et al. [2] who used it for free surface flows in forming processes. The pseudo-solid has never been used in FSI. It provides physics-based rules for deforming the fluid domain given the structural displacements of the solid-fluid interface.

$$\nabla \cdot \sigma_{ps}^l = 0 \quad (10)$$

$$\sigma_{ps}^l = \lambda_{ps} tr(E_{ps}^l) I + 2\mu_{ps} E_{ps}^l \quad (11)$$

$$E_{ps}^l = \frac{1}{2} \left[ h_{ps} + (h_{ps}^T)^T \right] \quad (12)$$

where $\lambda_{ps}$ and $\mu_{ps}$ are the pseudo-solid Lamé constants. These equations are supplemented by the following boundary conditions,

$$\chi_{ps} = 0 \quad \text{on} \ C_{\Gamma_f}(\Gamma) \quad (13)$$

These equations are solved in the fluid domain coupled to the Navier–Stokes equations and the structural equations at the interface location. Fluid-solid coupling is enforced via kinematic and equilibrium conditions:

$$\chi_{ps} = \chi_s \quad \text{on} \ \Gamma_I \quad (14)$$

$$\mathbf{u}_f = \mathbf{u}_s = 0 \quad \text{on} \ \Gamma_I \quad (15)$$

$$\sigma_c \cdot \mathbf{n}_s + \sigma_f \cdot \mathbf{n}_f = 0 \quad \text{on} \ \Gamma_I \quad (16)$$

where $\mathbf{n}_s$ is the outward unit normal to the solid at the solid-fluid interface in the deformed configuration, $\mathbf{n}_f = -\mathbf{n}_s$, and $\mathbf{u}_s$ is the solid velocity. For steady state problems, the interface velocity is prescribed to be zero. The stresses $\sigma$ are Cauchy stresses.

### 3 Solution strategy

The monolithic solution strategy couples all degrees of freedom: velocities, pressure, along with pseudo-solid and structural displacements. This approach requires a fully coupled-implicit treatment of all boundary conditions and interface loadings. This is illustrated in matrix form on Fig. 1(b). We make 3 key observations:

1. The pseudo-solid boundary displacement is set equal to the structural boundary displacements $\chi_{ps} = \chi_s$ (row 6 of the matrix on Fig. 1(b)).
2. Linearization of the pseudo-solid equations must account for all implicit dependencies to achieve quadratic convergence of Newton’s method [3].
3. The fluid loads are applied to the structure by treating nodal reactions as implicit unknowns (an implicit version of the reaction method [4]; rows 4 and 8 of the matrix on Fig. 1(b) [3]).
These operations are implemented simply and in a straightforward manner through interface elements enforcing relations (14) and (15), see Fig. 1(a). The interface element on the left enforces continuity of the displacements. The one on the right guarantees equilibrium of fluid and solid forces. The resulting global system matrix, shown on Fig. 1(b), illustrates the coupling between the various unknowns. In this figure, $u_{f}^{int}$, $r_{f}^{int}$ are the fluid velocity and reactions on the interface, $\chi_{ps}^{int}$, $\chi_{s}^{int}$ are the pseudo-solid and solid displacements along the interface and $u_{s}^{int}$, $r_{s}^{int}$ are the solid velocities and reactions on the interface. The shaded blocks represent contributions from the various weak forms. Identity matrices indicate that continuity of unknowns is enforced in strong form at the interface. The proposed monolithic formulation leads to quadratic convergence of Newton’s method. This is achieved at the cost of an increase in the number of unknowns and size of global matrix.
compared to a decoupled treatment of the overall system of equations. This is largely compensated by the significant reduction in the number of Newton iterations and CPU time savings.

4 Adaptive finite element procedure

The velocity and displacement fields are discretized using 6-noded quadratic elements. Fluid pressure is discretized by piecewise linear continuous functions. The equations are linearized by Newton’s method, assembled in a skyline structure and solved by Gaussian elimination. Error estimates are obtained for all solution fields (velocity, pressure, solid and pseudo-solid displacements) by a Zhu-Zienkiewicz error estimator [5, 6].

5 Numerical results

We present results for a flat plate in a cross flow and that for a flexible hydrofoil at an angle of attack of 6 degrees. We then apply our approach to fluid-structure interactions of a generic simplified model of an aneurysm and of an eyeball with abnormal inner pressure.

5.1 Flat plate in crossflow

The flow is horizontal from left to right (see Fig. 2(a)). The fluid is characterized by its density $\rho_f = 1$ and viscosity $\mu_f = 0.002$. This case corresponds to a flow at $Re = 500$ based on total plate height $H = 1$, fluid velocity $U = 1$ and viscosity; a material Young modulus ranging from 330 to $\infty$ and a Poisson coefficient $\nu = 0.25$ for flexible cases. The computational domain extends a distance of $4H$ upstream of the plate, $8H$ downstream, and $4H$ above and below. Only half of the domain is considered so that symmetry conditions are applied on the bottom of the domain. This case is interesting as it provides large deformations of the fluid domain. Also, classical hydrodynamic theory predicts that a moving rigid object experiences a drag proportional to the square of its speed. However, if the object is flexible, its flow-induced deformation results in reduced drag compared to the rigid configuration [7]. In ref. [7], this configuration is investigated experimentally and potential theory is used as a model. This fluid-structure interaction case has also been investigated by Oden [8] and Moller and Lund [9]. The usual assumption is that deflections of the plate are small compared to the plate height since small strain theory is usually invoked [8]. The only exception is the work of ref. [9]. In both works mesh cell distortions become so large as to prevent convergence of the solvers [9, 8]. Our formulation can reach much larger total displacements than previous approaches because of reduced mesh cell distortion in the fluid domain. Notice how regular fluid mesh cells are on the final deformed configuration shown on Fig. 2(b). From Figure 2(a), we can observe that as the Young modulus of the structure decreases, the plate tends to flex more and more in
an attempt to align itself with the flow. The Drag coefficient \( C_D = 2F/(\rho_f U^2 H) \) as a function of Young’s modulus is presented on Table 1. Results show that loads on the structures decrease with the Young modulus. This confirms the trend observed by Alben et al. [7] that as the plate deforms, its drag is reduced because its deformed configuration is more hydrodynamic.

5.2 Flexible hydrofoil at an angle of attack of 6°

We now study the effect of flexibility on the lift and drag of an Eppler 817 hydrofoil. We set density to \( \rho_f = 1 \) and viscosity \( \mu_f = 0.001 \) which case corresponds to \( R_e = 1000 \) based on the profile chord length \( C = 1 \), a free stream fluid velocity \( U = 1 \) and viscosity; the Young modulus \( E \) is varied from 200 to \( \infty \) and the Poisson coefficient is equal to \( \nu = 0.25 \). The computational domain extends a distance of \( 5C \) upstream of the foil, \( 4C \) downstream, and \( 5C \) above and below.
Table 1: Flat plate: Drag and top point location as a function of the Young modulus.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$C_D$</th>
<th>$X_{top}$</th>
<th>$Y_{top}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>330</td>
<td>0.605</td>
<td>0.35</td>
<td>0.34</td>
</tr>
<tr>
<td>1000</td>
<td>0.833</td>
<td>0.27</td>
<td>0.41</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.225</td>
<td>0.00</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 2: Hydrofoil: Drag and Lift coefficients as a function of the Young modulus.

<table>
<thead>
<tr>
<th>$E$</th>
<th>$C_D$</th>
<th>$C_L$</th>
<th>$E$</th>
<th>$C_D$</th>
<th>$C_L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>.143</td>
<td>.0321</td>
<td>2000</td>
<td>.122</td>
<td>.0424</td>
</tr>
<tr>
<td>300</td>
<td>.132</td>
<td>-.0636</td>
<td>6000</td>
<td>.124</td>
<td>.1944</td>
</tr>
<tr>
<td>600</td>
<td>.122</td>
<td>-.0694</td>
<td>$\infty$</td>
<td>.130</td>
<td>.3521</td>
</tr>
</tbody>
</table>

The flow is horizontal from left to right. For illustrative purpose, an angle of attack of 6 degrees has been imposed to show that important deformations of the foil can be computed (Figure 4(b)). Figure 4(b) shows an adapted fluid-structure mesh. Note that a pressure trough causes the concavity observed on the suction side. This is confirmed by the pressure field shown on figure 4(c). Also figure 4(d) shows that vorticity is produced on the upper side due to the concavity. The vertical solid and pseudo-solid displacements are shown on figure 4(e) and illustrate the fact that a vertical outer deflection is supported by the structure on the two sides. The deformed geometry appears less hydrodynamic than its rigid counterpart. The foil deforms to more or less align itself with the flow. Lift and drag coefficients are defined as $C_L = \frac{2F_y}{(\rho_f U^2 C)}$ and $C_D = \frac{2F_x}{(\rho_f U^2 C)}$ and are given in Table 2. We observe that as the Young modulus decreases, the Drag coefficient tends to decrease till $E = 600$ and increases as the Young modulus is further decreased. However, at the same time, a dramatic drop in Lift is observed revealing that the structure tends to align itself with the flow.

For extremely flexible materials ($E < 600$), the effect of pressure is much more pronounced. Figure 4(f) clearly shows the swelling of the cavity where a zero pressure is imposed. The two pressure minima on either sides of the shape cause the structure to inflate. For $E = 300$, the profile resembles to those observed for the more rigid structures shown on 4(b). However, for $E = 200$, the structure tends to become symmetric. The swelling induces an increase of the Drag coefficient as is depicted on Table 2.
5.3 A simplified model of an aneurysm

The behavior of a developed aneurysm is of primary importance to surgeons. Our model while simplified allows to draw conclusions with respect to stresses
\[ R_e = 100. \]
\[ E = (1 + x^2) \]

Figure 5: Cavity placed in a flexible channel.

Figure 6: Adapted deformed mesh.

and deformations of the wall. Geometry and boundary conditions are described on Fig. 5(a). The cavity represents the aneurysm. The Reynolds number of the flow is 100. The variable Young modulus equal to \( E = E_0 (1 + x^2) \) is used to reproduce a weakened arterial wall near the aneurysm. Note that in practice the mean static pressure is greater than the dynamic pressure. This results in numerical difficulties. Instead of using boundary conditions to set the pressure level, we use the reaction method to impose the hydrostatic pressure. The displacements at both ends are set to 0. Fig. 6 shows the deformed adapted mesh in the fluid and solid regions. It contains 110603 nodes. Even at such a low value of the Reynolds number, a recirculation zone occurs in the cavity as can be observed on Fig. 5(b). This recirculation zone induces a stress in the direction opposed to that of the flow. This explains why the cavity tends to lean backwards to the left. Another interesting observation is that edges of the aneurysm are subject to important dynamic pressure forces (low upstream and high downstream). This positive pressure gradient does not suffice to counterbalance viscous shear stresses that provoke the deformation.

5.4 An axisymmetric model of the eyeball

The behavior of the eyeball with respect to inner pressure is a key parameter in the understanding of the glaucoma impact on the neural optic nerve. We have chosen
an abnormal inner pressure of 50 mmHg. Note that normal inner pressure ranges from 12 to 20 mmHg. Geometry and boundary conditions are described on Fig. 7. In this case, there is nearly no flow in the eye and we have imposed zero flow. Meanwhile, a fluid domain has been modeled in order to impose inner pressure with the implicit reaction method as well as to ensure the following pressure condition. Fig. 8(a) shows the deformed adapted mesh in the fluid and solid regions. It contains 31174 nodes. The quadratic norm of the displacements has been plotted on Fig. 8(b) for the Lamina Cribrosa layer which is situated between the optic nerve and the Retina. We observe shear and radial compression of this layer. This is due to the difference in Young moduli between the Sclera (outer part of the eyeball) and the other tissues principally. Studying further the differences between a normal inner pressure in this case would allow to draw conclusions with respect to abnormal deformations of the optic nerve.
6 Conclusions

We have developed a general tightly fully coupled monolithic formulation for computing the interactions between an incompressible flow and a hyperelastic solid. A variational formulation of the problem is developed that ensures continuity of interface tractions and velocities. The formulation uses frames of reference that are natural to the fluid flow (Eulerian) and structural behavior (Lagrangian). The pseudo-solid formulation is used to provide control of mesh distortion during the computations. Several pseudo-solid materials and material laws have been developed which allow a capability to treat cases with very large displacements. The physics of the coupled fluid-structure interaction problem are obtained with high accuracy due to the adaptive procedure. It was successfully applied to cases of increasing complexity confirming the viability of the proposed approach.

Acknowledgements

This work was sponsored in part by NSERC (Government of Canada), the Canada Research Chair Program (Government of Canada), and by FQRNT (Government of Québec).

References