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Application of the ALE approach

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Abstract

A computational method is developed to solve the coupled fluid-structure interaction problem, where the viscous incompressible fluid and a rigid body-spring system interact with each other. In order to incorporate the effect of the moving surface of the rigid body, the arbitrary Lagrangian-Eulerian formulation is employed as the basis of the finite element spatial discretization. The predictor-corrector method is then used for the time integration of the equations of the motion.

1 Introdution

Many problems cannot be treated effectively with Lagrangian meshes. When the material is severely deformed, Lagrangian elements become similarly distorted since they deform with the material. The approximation accuracy of the elements then deteriorates, particularly for higher order elements. In some problems, Lagrangian methods are totally inappropriate. For example, in fluid mechanics with high velocity flows, interest is usually focused on a particular spatial subdomain, such as the domain around an airfoil. Similarly, the modeling of processes such as extrusion involve fixed spatial domains through which the material flows. These types of problems are more suited to Eulerian elements. In Eulerian finite elements, the elements are fixed in space and material convects through the elements. Eulerian finite elements thus undergo no distorsion due to material motion.

Unfortunately, the treatment of moving boundaries and interfaces is difficult with Eulerian elements. Therefore, hybrid techniques, which combine the advantages of Eulerian and Lagrangian methods, have been developed. These are called ALE methods: Arbitrary Lagrangian Eulerian. As the name suggests, ALE descriptions are arbitrary combinations of the Lagrangian and Eulerian descriptions. The word arbitrary here refers to the fact that thecombinations are specified by the user through the selection of a mesh motion.

2 Material motion and mesh displacement, velocity and acceleration

In ALE method, both the motions of the mesh and the material must be described. The motion of the material is described by

$$y = \Phi(x, t), \tag{1}$$

where x are the material coordinates. The function $\Phi(x, t)$ maps the body from the initial configuration Ω_0 to the current or spatial configuration Ω (*material motion*). Now we consider another reference domain $\hat{\Omega}$ (*referential domain* or *the ALE domain*). The initial values of the position of particles are denoted by χ, so

$$\chi = \Phi(x, 0). \tag{2}$$

The coordinates χ are called the referential or ALE coordinates. In most cases $\Phi(x, 0) = x$, so $\chi(x, 0) = x$. The referential domain $\hat{\Omega}$ is used to describe the motion of the mesh independent of the motion of the material. In the implementation, the domain $\hat{\Omega}$ is used to construct the initial mesh.

The motion of the mesh is described by

$$y = \hat{\Phi}(\chi, t). \tag{3}$$

This map $\hat{\Phi}$ plays a crucial role in the ALE finite formulation. Points χ in the ALE domain $\hat{\Omega}$ are mapped to points y in the spatial domain Ω, via this map.

As is apparent from eqs. (1) and (3), we can relate the ALE coordinates to the material coordinate by a composition of function. This relation has form

$$\chi = \hat{\Phi}^{-1}(y, t) = \hat{\Phi}^{-1}(\Phi(x, t), t) = \Psi(x, t). \tag{4}$$

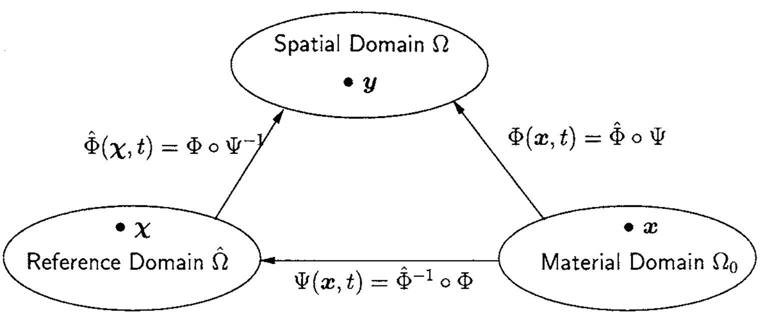

Figure 1: Maps between Lagrangian, Eulerian and ALE domains

We will now define the displacement, velocity and acceleration of the mesh motion. The *mesh velocity* $\hat{\boldsymbol{u}}$ is defined by

$$\hat{\boldsymbol{u}}(\chi, t) = \boldsymbol{y} - \chi = \hat{\Phi}(\chi, t) - \chi. \tag{5}$$

Note the similarly of the above definition to the definition of material displacement, which is $\boldsymbol{u} = \boldsymbol{y} - \boldsymbol{x}$ (the material coordinate has been replaced by the ALE referential coordinate). The *mesh velocity* is defined analogously to the material velocity and it holds

$$\hat{\boldsymbol{v}}(\chi, t) = \frac{\partial \hat{\Phi}(\chi, t)}{\partial t} \equiv \left. \frac{\partial \hat{\Phi}}{\partial t} \right|_{\chi}, \tag{6}$$

where the ALE coordinate χ is fixed. And finally the *mesh acceleration* is given by

$$\hat{\boldsymbol{a}} = \frac{\partial \hat{\boldsymbol{v}}(\chi, t)}{\partial t} = \left. \frac{\partial^2 \hat{\boldsymbol{u}}(\chi, t)}{\partial t^2} \right|_{\chi}. \tag{7}$$

Neither the mesh acceleration nor the mesh velocity have any physical meaning in an ALE mesh which is not Lagrangian. When the mesh is Lagrangian, they correspond to the material velocity and acceleration.

3 Material time derivative and convective velocity

In ALE descriptions, fields are usually expressed as functions of ALE coordinates χ and time t. The material derivative must then be obtained by the chain rule, similar to the process used in an Eulerian description. Consider a specific function $f(\chi, t)$. Using the chain rule gives

$$\frac{Df}{Dt} \equiv \dot{f}(\chi, t) = \frac{\partial f(\chi, t)}{\partial t} + \frac{\partial f(\chi, t)}{\partial \chi_i} \frac{\partial \Psi_i(\boldsymbol{x}, t)}{\partial t} = \left. \frac{\partial f}{\partial t} \right|_{\chi} + \frac{\partial f}{\partial \chi_i} w_i, \tag{8}$$

where w_i is *the referential particle velocity* and is defined as

$$w_i = \frac{\partial \Psi_i(\boldsymbol{x}, t)}{\partial t} = \left. \frac{\partial \chi_i}{\partial t} \right|_{\chi}. \tag{9}$$

In the praxis the ALE field variables are often treated as functions of the material coordinates $\boldsymbol{x}$ and time t. Hence, it is convenient to develop expressions for the material time derivative in terms of the spatial gradient. For the material velocity holds

$$v_i = \frac{\partial \Phi_i(\boldsymbol{x}, t)}{\partial t} = \hat{v}_j + \frac{\partial y_i}{\partial \chi_i} \frac{\partial \chi_i}{\partial t}. \tag{10}$$

Now we can define *the convective velocity* **c** as the diference between the material and mesh velocities. It can be shown, that

$$c_i = v_i - \hat{v}_i = \frac{\partial y_i(\chi, t)}{\partial \chi_j} w_j. \tag{11}$$

This relationship between the convective velocity $\boldsymbol{c}$, material velocity $\boldsymbol{v}$, mesh velocity $\hat{\boldsymbol{v}}$ and the referential velocity $\boldsymbol{w}$ will be used frequently in the ALE formulation.

Because of eqs. (8), (9) and (11) we can write expression for the material time derivative with a spatial gradient

$$\frac{Df}{Dt} = \left.\frac{\partial f}{\partial t}\right|_{\chi} + \frac{\partial f}{\partial y_j}\frac{\partial y_j}{\partial \chi_i}w_i = \left.\frac{\partial f}{\partial t}\right|_{\chi} + \frac{\partial f}{\partial y_j}c_j. \tag{12}$$

4 Problem statement

Figure 2 shows a schematic desription of the interaction problem, where $\Omega_B(t)$ is the domain occupied by the moving rigid body (S is a center of gravity), $\Omega_F(t)$ is the moving spatial domain upon which the fluid motion is described. $\Gamma_B(t)$ is the interface between $\Omega_B(t)$ and $\Omega_F(t)$. As the rigid body $\Omega_B(t)$ changes its position, the interface $\Gamma_B(t)$ moves accordingly. Providing that we can specify in some way the distribution of the mesh velocity $\hat{v}_i(\boldsymbol{x}$,t) in $\Omega_F(t)$ in accordance with the motion of $\Omega_B(t)$, we can employ the following ALE description of the Navier-Stokes and continuity equations of the fluid motion

$$\varrho\frac{\partial v_i}{\partial t} + \varrho(v_i - \hat{v}_i)\frac{\partial v_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j}\right) + \varrho f_i, \tag{13}$$

$$\frac{\partial v_i}{\partial x_i} = 0, \qquad \text{in } \Omega_F(t) \tag{14}$$

where v_i is the the material velocity vector of the fluid, $(v_i - \hat{v}_i)$ is the convective velocity ($\hat{v}_i$ – mesh velocity), ϱ is the density, x_i are Euler's (spatial) coordinates, p is the pressure, f_i is the specific body force vector, μ is the dynamic viscosity of the fluid.

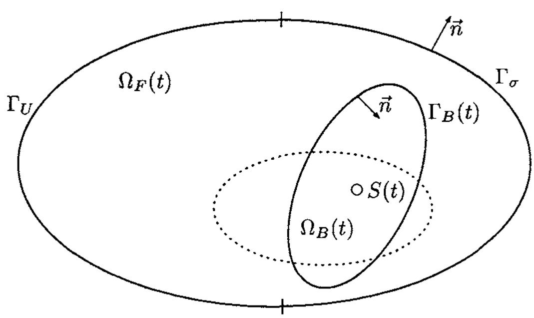

Figure 2: Scheme of the problem domain

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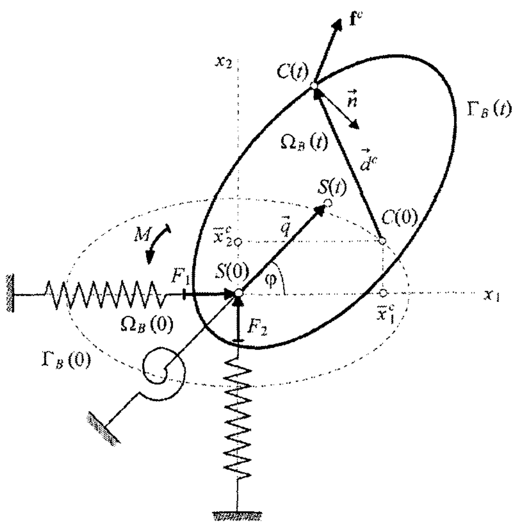

Figure 3: Displacement and forces on the body

The Dirichlet a Neumann boundary conditions

$$v_i = g_i, \quad \text{on } \Gamma_U; \qquad \sigma_i = \tau_{ij} n_j = h_i, \quad \text{on } \Gamma_\sigma. \tag{15}$$

Boundary conditions for the mesh velocity

$$\hat{v}_i = 0, \quad \text{on } \Gamma_U \cup \Gamma_\sigma; \qquad \hat{v}_i = v_i^B \quad \text{on } \Gamma_B(t). \tag{16}$$

5 The equation of motion of the rigid body

We consider a general planar motion of a single rigid body (3 DOFs), of which motion is described by three displacement components defined at the center of gravity S

$$\boldsymbol{q} = [q_1 \quad q_2 \quad \varphi]^T, \tag{17}$$

In this case the equations of motion of the rigid body are written as

$$\boldsymbol{M}\ddot{\boldsymbol{q}}(t) + \boldsymbol{B}\dot{\boldsymbol{q}}(t) + \boldsymbol{K}\boldsymbol{q}(t) = \boldsymbol{f}_B(t), \tag{18}$$

where $\boldsymbol{M}$, $\boldsymbol{B}$ and $\boldsymbol{K}$ are the mass matrix, the damping matrix and the stiffness matrix, the vector $\boldsymbol{f}_B$ contains resultants of the surface traction.

6 Spatial discretization

Finite element discretization over the moving spatial domain of the fluid $\Omega_F(t)$ leads to the following equations

$$\boldsymbol{C}\boldsymbol{a} + \boldsymbol{D}(\boldsymbol{v} - \hat{\boldsymbol{v}})\boldsymbol{v} - \boldsymbol{G}\boldsymbol{p} = \boldsymbol{f} \quad \text{and} \quad \boldsymbol{G}^T\boldsymbol{v} = \boldsymbol{0}, \tag{19}$$

These equations can be represented as

$$\begin{bmatrix} C^{\alpha\alpha} & C^{\alpha\beta} & C^{\alpha\gamma} \\ C^{\beta\alpha} & C^{\beta\beta} & C^{\beta\gamma} \\ C^{\gamma\alpha} & C^{\gamma\beta} & C^{\gamma\gamma} \end{bmatrix} \begin{bmatrix} a^{\alpha} \\ \bar{a}^{\beta} \\ a^{\gamma} \end{bmatrix} + $$

$$+ \begin{bmatrix} D^{\alpha\alpha} & D^{\alpha\beta} & D^{\alpha\gamma} \\ D^{\beta\alpha} & D^{\beta\beta} & D^{\beta\gamma} \\ D^{\gamma\alpha} & D^{\gamma\beta} & D^{\gamma\gamma} \end{bmatrix} \begin{bmatrix} v^{\alpha} \\ \bar{v}^{\beta} \\ v^{\gamma} \end{bmatrix} - \begin{bmatrix} G^{\alpha} \\ G^{\beta} \\ G^{\gamma} \end{bmatrix} p = \begin{bmatrix} \bar{f}^{\alpha} \\ f^{\beta} \\ f^{\gamma} \end{bmatrix},$$

$$\begin{bmatrix} G^{\alpha T} & G^{\beta T} & G^{\gamma T} \end{bmatrix} \begin{bmatrix} v^{\alpha} \\ \bar{v}^{\beta} \\ v^{\gamma} \end{bmatrix} = 0, \tag{20}$$

where matrices and vectors were divided into three parts: $^{\alpha}$-part is associated with $\Omega_F(t)$ or Γ_σ, $^{\beta}$-part with boundary Γ_U and $^{\gamma}$-part with $\Gamma_B(t)$. The overbar ($\bar{\ }$) denotes prescribed values.

Relations between the rigid-body surface $\Gamma_B(t)$ variables and the body freedoms are given by compatibility conditions

$$v^{\gamma} = T^T \dot{q}, \quad a^{\gamma} = T^T \ddot{q} + A\dot{\varphi} \tag{21}$$

and equilibrium condition

$$f_B + T f^{\gamma} = 0. \tag{22}$$

The transformation matrices T and A is derived from geometrical relations between the center of gravity S of the rigid body and the nodal coordinates of each node on $\Gamma_B(t)$.

The mesh velocity vector $\hat{v}$ has to satisfy the boundary conditions

$$\hat{v} = v^{\gamma} = T^T \dot{q}, \quad \text{on } \Gamma_B(t), \qquad \hat{v} = 0, \quad \text{on } \Gamma_\sigma \cup \Gamma_U, \tag{23}$$

and the mesh velocity between these two boundaries is given by simple linear function.

7 Derivation of the solution procedure

To eliminate the nodal components on the rigid body surface $\Gamma_B(t)$ we can use compatibility conditions (21) and also we can remove the second row from (20), we obtain

$$\begin{bmatrix} C^{\alpha\alpha} & C^{\alpha\gamma} \\ C^{\gamma\alpha} & C^{\gamma\gamma} \end{bmatrix} \begin{bmatrix} a^{\alpha} \\ T^T \ddot{q} + A\dot{\varphi}^2 \end{bmatrix} + \begin{bmatrix} D^{\alpha\alpha} & D^{\alpha\gamma} \\ D^{\gamma\alpha} & D^{\gamma\gamma} \end{bmatrix} \begin{bmatrix} v^{\alpha} \\ T^T \dot{q} \end{bmatrix} -$$

$$- \begin{bmatrix} G^{\alpha} \\ G^{\gamma} \end{bmatrix} p = \begin{bmatrix} \bar{f}^{\alpha} \\ f^{\gamma} \end{bmatrix} - \begin{bmatrix} C^{\alpha\beta} \\ C^{\gamma\beta} \end{bmatrix} \bar{a}^{\beta} - \begin{bmatrix} D^{\alpha\beta} \\ D^{\gamma\beta} \end{bmatrix} \bar{v}^{\beta}. \tag{24}$$

Because of equilibrium condition (22), we can obtain $\boldsymbol{f}_B$ by expressing of force $\boldsymbol{f}^\gamma$ from second row of (24). The equation (18) of motion of the rigid body can be then written as

$$\boldsymbol{M}^*\ddot{\boldsymbol{q}} + \boldsymbol{B}\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = -\boldsymbol{T}\left(\begin{bmatrix} \boldsymbol{C}^{\gamma\alpha} & \boldsymbol{C}^{\gamma\beta} & \boldsymbol{C}^{\gamma\gamma} \end{bmatrix}\begin{bmatrix} \boldsymbol{a}^\alpha \\ \bar{\boldsymbol{a}}^\beta \\ \boldsymbol{A}\dot{\varphi}^2 \end{bmatrix} + \right.$$
$$\left. + \begin{bmatrix} \boldsymbol{D}^{\gamma\alpha} & \boldsymbol{D}^{\gamma\beta} & \boldsymbol{D}^{\gamma\gamma} \end{bmatrix}\begin{bmatrix} \boldsymbol{v}^\alpha \\ \bar{\boldsymbol{v}}^\beta \\ \boldsymbol{v}^\gamma \end{bmatrix} - \boldsymbol{G}^\gamma \boldsymbol{p}\right), \quad (25)$$

where

$$\boldsymbol{M}^* = \boldsymbol{M} + \boldsymbol{T}\boldsymbol{C}^{\gamma\gamma}\boldsymbol{T}^T. \quad (26)$$

8 The predictor-multicorrector method

The time integration algorithm to solve first row of (24) and (25) is based on the predictor-multicorrector method. The solution procedure using this algorithm is summarised as follows for one step integration from $t = t_n$ to $t = t_{n+1} = t_n + \Delta t$:
Firts phase: predictor $(i = 0)$

$$[\text{fluid}] \begin{cases} \boldsymbol{a}_{n+1}^{\alpha(i)} = \boldsymbol{0}, \\ \boldsymbol{v}_{n+1}^{\alpha(i)} = \boldsymbol{v}_n^\alpha + \Delta t(1-\gamma_v)\boldsymbol{a}_n^\alpha, \\ \boldsymbol{p}_{n+1}^{(i)} = \boldsymbol{p}_n. \end{cases} \quad (27)$$

$$[\text{rigid body}] \begin{cases} \ddot{\boldsymbol{q}}_{n+1}^{(i)} = \boldsymbol{0}, \\ \dot{\boldsymbol{q}}_{n+1}^{(i)} = \boldsymbol{q}_n + \Delta t(1-\gamma)\ddot{\boldsymbol{q}}_n, \\ \boldsymbol{q}_{n+1}^{(i)} = \boldsymbol{q}_n + \Delta t\dot{\boldsymbol{q}}_n + \frac{1}{2}(\Delta t)^2(1-2\beta)\ddot{\boldsymbol{q}}_n. \end{cases} \quad (28)$$

Second phase: solution $(0 \le i \le I-1)$
The residuals of the first row (24) and (25) at the i-th stage of approximation at time t_{n+1} are written respectively as

$$\boldsymbol{R}^{\alpha(i)} = \bar{\boldsymbol{f}}^\alpha - \begin{bmatrix} \boldsymbol{C}^{\alpha\alpha} & \boldsymbol{C}^{\alpha\beta} & \boldsymbol{C}^{\alpha\gamma} \end{bmatrix}^{(i)}\begin{bmatrix} \boldsymbol{a}^\alpha \\ \bar{\boldsymbol{a}}^\beta \\ \boldsymbol{T}^T\ddot{\boldsymbol{q}} + \boldsymbol{A}\dot{\varphi}^2 \end{bmatrix}^{(i)} -$$
$$- \begin{bmatrix} \boldsymbol{D}^{\alpha\alpha} & \boldsymbol{D}^{\alpha\beta} & \boldsymbol{D}^{\alpha\gamma} \end{bmatrix}^{(i)}\begin{bmatrix} \boldsymbol{v}^\alpha \\ \bar{\boldsymbol{v}}^\beta \\ \boldsymbol{T}^T\dot{\boldsymbol{q}} \end{bmatrix}^{(i)} + \boldsymbol{G}^{\alpha(i)}\boldsymbol{p}^{(i)}, \quad (29)$$

$$
r^{(i)} = -M^{*(i)}\ddot{q}^{(i)} - B\dot{q}^{(i)} - Kq^{(i)} - T^{(i)}\left(\begin{bmatrix} C^{\gamma\alpha} & C^{\gamma\beta} & C^{\gamma\gamma}\end{bmatrix}^{(i)}\begin{bmatrix} a^{\alpha} \\ \bar{a}^{\beta} \\ A\dot{\varphi}^2 \end{bmatrix}^{(i)} + \right.
$$
$$
\left. + \begin{bmatrix} D^{\gamma\alpha} & D^{\gamma\beta} & D^{\gamma\gamma}\end{bmatrix}^{(i)}\begin{bmatrix} v^{\alpha} \\ \bar{v}^{\beta} \\ T^T\dot{q} \end{bmatrix}^{(i)} - G^{\gamma}p^{(i)}\right). \quad (30)
$$

Now we evaluate the acceleration increments and the pressure increment in order to update the correctors as follows:

$$
\bar{C}^{\alpha(i)}\Delta a^{\alpha(i)} - G^{\alpha(i)}\Delta p^{(i)} \equiv \bar{C}^{\alpha(i)}\Delta a^{\alpha*(i)} = R^{\alpha(i)}, \quad (31)
$$
$$
\bar{M}^{*(i)}\Delta\ddot{q}^{(i)} - T^{(i)}G^{\gamma(i)}\Delta p^{(i)} \equiv \bar{M}^{*(i)}\Delta\ddot{q}^{*(i)} = r^{(i)}, \quad (32)
$$

where $\bar{C}^{\alpha(i)}$ denotes the lumped counterpart of $C^{\alpha\alpha}$, and

$$
\bar{M}^{*(i)} = M^* + \Delta t\gamma B + (\Delta t)^2\beta K. \quad (33)
$$

The increments $\Delta a^{\alpha(i)}$ and $\Delta\ddot{q}^{(i)}$ can be then expressed as

$$
\Delta a^{\alpha(i)} = \Delta a^{\alpha*(i)} + \bar{C}^{\alpha(i)-1}G^{\alpha(i)}\Delta p^{(i)}, \quad (34)
$$
$$
\Delta\ddot{q}^{(i)} = \Delta\ddot{q}^{*(i)} + \bar{M}^{*(i)-1}T^{(i)}G^{\gamma(i)}\Delta p^{(i)}. \quad (35)
$$

The remaining and final problem is the determination of pressure increment $\Delta p^{(i+1)}$. We can obtain it as a solution of

$$
P^{(i)}\Delta p^{(i+1)} = -\begin{bmatrix} G^{\alpha T} & G^{\beta T} & G^{\gamma T}\end{bmatrix}^{(i)}\begin{bmatrix} v^{\alpha(i)} + \gamma\Delta t\Delta a^{\alpha*(i)} \\ \bar{v}^{\beta} \\ {T^{(i)}}^T(\dot{q}^{(i)} + \gamma\Delta t\Delta\ddot{q}^{*(i)}) \end{bmatrix}, \quad (36)
$$

where

$$
P^{(i)} = \gamma_v\Delta t G^{\alpha(i)}\bar{C}^{\alpha(i)-1}{G^{\gamma(i)}}^T + \gamma\Delta t {G^{\gamma(i)}}^T {T^{(i)}}^T \bar{M}^{*(i)-1}T^{(i)}G^{\gamma(i)}. \quad (37)
$$

Third phase: corrector $(0 \le i \le I-1)$

$$
[\text{fluid}]\begin{cases} a_{n+1}^{\alpha(i+1)} = a_{n+1}^{\alpha(i)} + \Delta a^{\alpha(i)}, \\ v_{n+1}^{\alpha(i+1)} = v_{n+1}^{\alpha(i)} + \gamma_v\Delta t\Delta a^{\alpha(i)}, \\ p_{n+1}^{(i+1)} = p_{n+1}^{(i)} + \Delta p^{(i)}. \end{cases} \quad (38)
$$

$$
[\text{rigid body}]\begin{cases} \ddot{q}_{n+1}^{(i+1)} = \ddot{q}_{n+1}^{(i)} + \Delta\ddot{q}^{(i)}, \\ \dot{q}_{n+1}^{(i+1)} = \dot{q}_{n+1}^{(i)} + \gamma\Delta t\Delta\ddot{q}^{(i)}, \\ q_{n+1}^{(i+1)} = q_{n+1}^{(i)} + \beta(\Delta t)^2\Delta\ddot{q}^{(i)}. \end{cases} \quad (39)
$$

In the above, both the second phase and the third phase are integrated I times $(I \ge 2)$. The parameters γ_v, γ and β are governing the algorithmic damping and

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accuracy. Due to the stability condition, the following conditions are required for these parameters $\gamma_v \geq \frac{1}{2}, \gamma \geq \frac{1}{2}, \beta \geq \frac{1}{4}$.

9 Numerical results

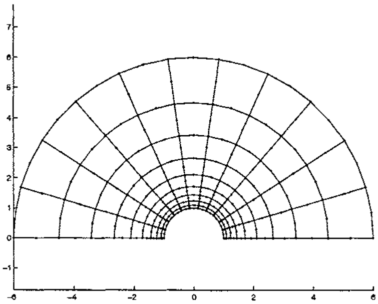

Figure 4: Original mesh

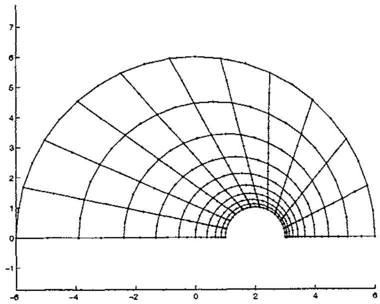

Figure 5: Mesh in the motion

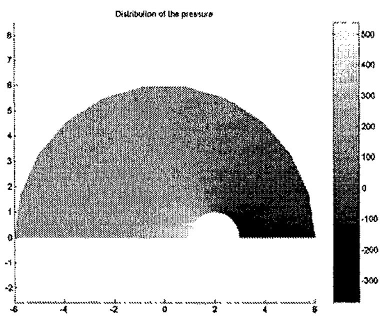

Figure 6: Distribution of the pressure

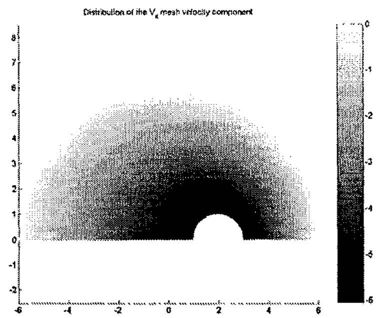

Figure 7: Mesh velocity – v_x distrib.

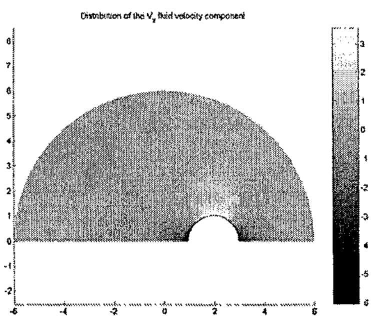

Figure 8: Fluid velocity – v_x distrib.

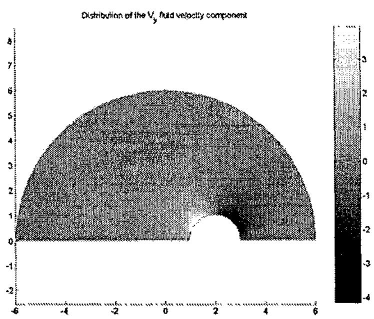

Figure 9: Fluid velocity – v_y distrib.

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10 Conclusion

We have created a computional model of free vibrations of a single circular cylinder in a viscous fluid, which fills a circular domain (see **fig. 4**). The motion of the cylinder has prescribed one degree of freedom in direction of the axis x. We have treated an axial-symetrical case, without spring damping.

On the **fig. 5** is shown a deformed mesh in the motion. The rest pictures show the distribution of the fluid velocity (**fig. 8**, **9**) and the pressure (**fig. 6**) and the distribution of the mesh velocity (**fig. 7**). They are caught at the same moment as **fig. 5**, when the cylinder goes back in direction $-x$.

We want later to add degree of freedom in direction of the axis y and the rotational degree of freedom. And finally we want to compute the damping for various kind of fluid environments in the future.

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