The use of trigonometric interpolation functions for vibration analysis of beam structures – bridging gap between FEM and exact formulations

S.M. Hashemi
Department of Mechanical, Aerospace & Industrial Engineering, Ryerson University, Canada

Abstract

The small linear harmonic oscillations of linearly elastic coupled beam structures are addressed. The Finite Element Method (FEM), conventional Hermit beam formulation, and the resulting element matrices are then briefly discussed. The so-called exact Dynamic Stiffness Matrix (DSM) formulation for the longitudinal vibration of axially loaded beams is presented. Finally, the Dynamic Finite Element (DFE) approach is introduced and its application to the axial vibration of beams is displayed. The comparison is made between the standard static and (frequency dependent) Dynamic beam shape functions. The DFE formulation, combines the generality of the FEM and the high precision provided by DSM methods. The weighting functions and shape functions are evaluated referring to the appropriate exact DSM formulation. The DFE approach can be advantageously extended to more complex cases which distinguishes this method from the DSM method.

1 Introduction

Many mechanical systems such as helicopter, turbine, compressor and propeller blades, spinning spacecraft, satellites, and also rotating shafts and linkages as well as various large terrestrial or aerospace structures, can be modelled as axially loaded slender coupled beams or beam assemblies. The dynamic modeling and the equations of motion for vibrations of various beam configurations, studied by different authors, have led to different approaches for the frequency calculation of these structures. Some good
literature surveys have been reported by researchers (see for example Reference [1]). One can also find in Reference [2] a review of several approximate methods such as Mykelstad method, the Galerkin method, the Rayleigh-Ritz method, the finite element method, etc., with a special regard to helicopter blades analysis. The classical Finite Element Method (FEM), where beam element matrices are evaluated from assumed fixed shape functions, has been used by numerous investigators [1, 3]. In FEM polynomial shape functions are often used which, in this case, results in approximate equations in the form of mass and static stiffness matrices. The Dynamic Stiffness Matrix (DSM) method probably offers a better alternative, particularly when higher frequencies and better accuracies of results are required. It relies on a single frequency dependent matrix which has both mass and stiffness properties. The use of a DSM in vibration analysis is well established [4, 5, 6]. Obviously the method gives more accurate results because it exploits the exact member theory. The matrix is obtained by directly solving the governing differential equation. That is why the results obtained by DSM are often justifiably called "exact" [6]. Other methods also exist in the literature which are more or less similar to those explained previously. Some of them were developed to treat a particular group of mechanical systems.

The main idea of this paper is to show that a deviation from the conventional FEM formulation would pay dividends if improved accuracy of results can be obtained by using shape functions other than polynomials. This is the case when the homogeneous solution of the pertinent differential equation is available for the development of each element matrices. For static analysis, it has been proven that the use of the homogeneous solution of the differential equation yields the exact stiffness matrix and load vector for an element [7]. A thorough investigation of the existing conventional FEM and alternative DSM methods in beam vibrations has helped to develop the idea of Dynamic (frequency dependent) Finite Element (DFE) approach. DFE can bridge the gap between the standard FEM and the exact DSM methods by advantageously exploiting the generality of FEM and the very precise frequency calculations provided by DSM approach. From this point of view, the method retains the physical aspect of the analytical or semi-analytical approach and the power of numerical methods. It has been shown that DFE is an efficient tool for handling periodical structures or systems composed of several identical substructures. In this case, all substructures have the same dynamic stiffness components and frequency characteristics so that increasing the number of substructures does not significantly increase the computation time.

2 Theory

The governing differential equations of motion for an axially loaded uniform beam with an open or closed thin-walled cross-section of the type shown in Figure 1 is presented in reference [8]. The Shear deformation, warping and rotary inertia effects are neglected. The mass axis and the elastic axis being separated by a distance \( x_0 \). In the right handed coordinate system shown in Figure 1 the elastic axis which is assumed to be coincident with the \( x \)-axis is permitted axial deformation \( u(x, t) \), flexural translation \( w(x, t) \) in the \( z \)-direction and torsional rotation \( \psi(x, t) \) about \( z \)-axis where \( x \) and \( t \) denote distance from the origin and time respectively. A constant axial load \( P \) is assumed to act through the centroid (mass centre) of the cross section. \( P \) is considered
Figure 1: Coordinate system and notations for coupled bending-torsional vibration of an axially loaded uniform beam element: $E_s$ shear centre; $G_s$ mass centre.

to be positive when tractional, as shown in the figure ($P$ can be positive or negative. Hence compression is also included). Thence, assuming sinusoidal variation of $w$ and $\psi$, with circular frequency $\omega$, 

$$u(x, t) = u(x)\sin(\omega t); \quad w(x, t) = w(x)\sin(\omega t); \quad \psi(x, t) = \psi(x)\sin(\omega t)$$ (1)

the differential equations governing the beam vibrations (see Figure 1) are [8]:

$$( -H_m u' )' - m\omega^2 u = 0$$ (2)

$$(H_f w'')'' - (Pw')' - m\omega^2 w + [(Px_\alpha \psi)' + mx_\alpha \omega^2 \psi] = 0$$ (3)

$$[-(H_t + P(I_\alpha / m)) \psi'']' - I_\alpha \omega^2 \psi + [(Px_\alpha w)' + mx_\alpha \omega^2 w] = 0$$ (4)

where $u(x)$, $w(x)$ and $\psi(x)$ are the amplitudes of the sinusoidally varying axial and vertical displacements and torsional rotation respectively. $H_m = EA$, $H_f = EI$ and $H_t = GJ$ are respectively the membrane, bending and torsional rigidity of the beam; $m = \rho A$ is the mass per unit length, $I_\alpha$ is the polar mass moment of inertia per unit length about the $x$-axis (i.e., an axis through the shear centre) and $t$ and primes denote time and differentiation with respect to position $x$, respectively. The appropriate boundary conditions are imposed at $x = 0, L$. For example:

Clamped: $x = 0; \quad u = w = w' = \psi = 0$,

Free: $x = L; \quad u' = w'' = w''' = \psi' = 0$, etc. The resultant loads are:

$$N(x) = H_m u' \quad T(x) = (-H_f w'')' + P(w' - x_\alpha \psi')$$

$$M_F(x) = -H_f w'' \quad M_T(x) = (H_t + P(I_\alpha / m)) \psi' - (Px_\alpha w)'$$

2.1 Exact dynamic stiffness matrix (DSM) formulation

In a large number of vibration problems, the pertinent governing differential equations intercorporate variable coefficients. In that case, it is impossible to find the exact
(or closed form solution) of these equations. For example, let us consider a spinning nonuniform coupled bending-torsion beam. An expedient way of dealing with such structure with variable parameters would be to break it down into several members, joined end to end, such that the element equations can all presumably have constant coefficients (i.e., the constituent members are considered to be straight, uniform, Bernoulli-Euler beams with Saint-Venant torsion theory, modelled by the eqns (2-4)). Within each element, in that case, the axial force is also taken as constant and equal to the mean value of the true centrifugal sectional force in that element. Consequently, it will be possible to derive the exact member DSM. Then, the overall stiffness matrix is found by assembling the member DSMs. It can be realized that the axial vibration of the beam is independent of the axial force. Moreover, since there is no coupling between the the axial and the lateral deformations, the membrane deformation and bending-torsion vibrations could be treated separately. In the following, the derivation of the exact DSM for the axial vibration of a prestressed beam, governed by eqn (2), is briefly discussed. Similarly, one could exploit the DSM approach to analyze the beam uncoupled bending, torsion, and coupled bending-torsion vibrations governed by eqns (3) and (4) [4, 5, 6].

2.1.1 Exact DSM for the axial vibration of beams

Assuming the the beam as the assembly of elements with homogeneous material and uniform geometry, eqn (2) can be written as

\[-H_m D^2 - m \omega^2 l^2 u = 0; \quad D = d/d\xi \quad \text{and} \quad \xi = x/l. \quad (5)\]

The solution of the differential eqn (5) is obtained as

\[u = < P(\xi) > * \{A\} \quad \text{where} \quad < P(\xi) > = \begin{pmatrix} \cos(\alpha \xi) \\ \sin(\alpha \xi) \end{pmatrix} \quad (6)\]

where \(A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\), \(\alpha^2 = \frac{\omega^2}{E}\), and \(l = x_{j+1} - x_j\). Then, the axial force is

\[N(\xi) = \frac{d}{d\xi} \begin{pmatrix} H_m/L \\ \xi \end{pmatrix} = \begin{pmatrix} H_m/L \end{pmatrix} < P(\xi),_\xi > * \{A\} \quad (7)\]

The end conditions for displacements and forces of the beam element are respectively given as follows.

at end 1 (i.e., at \(\xi = 0\)); \(u = u_1\) and at end 2 (i.e., at \(\xi = 1\)); \(u = u_2\)

at end 1 (i.e., at \(\xi = 0\)); \(N = -N_1\) and at end 2 (i.e., at \(\xi = 1\)); \(N = N_2\)

Substituting these boundary conditions in eqns (6) and (7)

\[\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} < P(\xi),_{\xi=0} > \\ < P(\xi),_{\xi=1} > \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{(i.e.,} \quad U_n = BA \quad (8)\]

Then, from eqns (6–8) we can write

\[\begin{pmatrix} N_1 \\ N_2 \end{pmatrix} = \begin{pmatrix} -(H_m/L) < P(\xi),_\xi >_{\xi=0} \\ (H_m/L) < P(\xi),_\xi >_{\xi=1} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \quad \text{(i.e.,} \quad F = DA \quad (9)\]
and eqns (8) and (9) lead to

\[
\begin{bmatrix}
N_1 \\
N_2
\end{bmatrix} = \begin{bmatrix}
k_{11} & k_{12} \\
sym. & k_{22}
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2
\end{bmatrix}
\quad \text{(i.e., } F = KU_n) \tag{10}
\]

Finally, the dynamic stiffness matrix can be written as

\[
k(\alpha) = \frac{H_m\alpha}{\sin(\alpha l)} \begin{bmatrix}
\cos(\alpha l) & -1 \\
-1 & \cos(\alpha l)
\end{bmatrix}
\quad \text{(i.e., } K = DB^{-1}) \tag{11}
\]

2.2 Weak integral formulation

The Galerkin weak form associated to eqns (2-4) can be written as:

\[
\mathcal{W} = \mathcal{W}_{INT} - \mathcal{W}_{EXT} = (\mathcal{W}_f + \mathcal{W}_t + \mathcal{W}_m) - \mathcal{W}_{EXT} = 0 \tag{12}
\]

where, for free vibration analysis, \( \mathcal{W}_{EXT} = 0 \). If the domain is discretized by a number of 2-node elements, then

\[
\mathcal{W}_{INT} = \sum_{k=1}^{N_{\text{E}}} \mathcal{W}^e \quad \text{where } \mathcal{W}^e = \mathcal{W}^e_f + \mathcal{W}^e_t + \mathcal{W}^e_m; \tag{13}
\]

and

\[
\mathcal{W}^e_m = \int_{x_j}^{x_{j+1}} \left\{ \delta u' (H_m) u' - \delta u (m\omega^2) u \right\} \, dx \tag{14}
\]

\[
\mathcal{W}^e_f = \int_{x_j}^{x_{j+1}} \left\{ \delta w'' (H_f) w'' + \delta w' (P) w' - \delta w (m\omega^2) w \\
- \delta w' (P x_\alpha) \psi' + \delta w (m x_\alpha \omega^2) \psi \right\} \, dx \tag{15}
\]

\[
\mathcal{W}^e_t = \int_{x_j}^{x_{j+1}} \left\{ \delta \psi' (H_t + P (I_\alpha/m)) \psi' - \delta \psi (I_\alpha \omega^2) \psi \\
- \delta \psi' (P x_\alpha) w' + \delta \psi (m x_\alpha \omega^2) w \right\} \, dx \tag{16}
\]

Here, \( u, w \) and \( \psi \) are solution functions and \( \delta u, \delta w \) and \( \delta \psi \) are test functions. Both quantities are defined in same approximation space. Each element is defined by nodes \( j, j + 1 \) with corresponding coordinates. For Clamped-Free boundary conditions: \( \delta u = \delta w = \delta w' = \delta \psi = 0 \) at \( x = 0 \), and force terms are zero at \( x = L \). The admissibility condition for finite element approximation is controlled by eqns (14-16).

### 2.2.1 Axial vibration of beams: exact dynamic finite element DFE

In this section, the application of the Dynamic Finite Element (DFE) procedure to the beam axial vibration is presented. The formulation presented here results in the same exact frequency dependent formulation and stiffness matrix as that obtained from the exact beam Dynamic Stiffness Matrix (DSM) formulation (see section 2.1.1). This procedure can easily be extended to develop the DFE formulation and the relevant matrices for the different cases such as pure torsion and bending, of axially loaded
uniform and tapered beams [9]. Consider again the axial vibration of as governed by Differential (i.e., strong form) eqn (2) and appropriate boundary conditions at \( x = 0 \) and \( x = L \). The Weak integral form of the governing equation can then be written as:

\[
\int_{0}^{L} \delta u\{-(H_mu,x)_{x} - m\omega^2u\}dx = 0
\]

(17)

where \( u \) is the solution function and \( \delta u \) is the Galerkin type test function. Integrating by parts permits to diminish the derivatives order and the Galerkin type weak form (VWP) of eqn (2) is then written as:

\[
W = W_{INT} - W_{EXT} = 0; \text{ where } W_{INT} = \int_{0}^{L} \{ \delta u,_{x}H_mu,_{x} - \delta um\omega^2u \}dx
\]

(18)

and \( W_{EXT} = 0 \) for free vibrations. \( W_{INT} \) is then discretized by Finite Elements with constant \( E, \rho \) and \( A \).

\[
W_{INT} = \sum_{e=1}^{NE} W^e; \text{ where } W^e = \int_{x_j}^{x_{j+1}} \{ \delta u,_{x}H_mu,_{x} - \delta um\omega^2u \}dx
\]

(19)

After integration by parts, and assuming a uniform homogeneous element, the expression 19 can be then written in the following form:

\[
W^e = -\int_{x_j}^{x_{j+1}} \{ \delta u,_{xx}H_m + \delta um\omega^2u \}udx + [\delta u,_{x}H_mu]^{x_{j+1}}_{x_j}
\]

(20)

Assuming \( \delta u(x) = < P \{ \delta A \}, u(x) = < P \{ A \} \) and \( < P = \cos \alpha x \frac{\sin \alpha x}{\alpha} > \), the integral expression in eqn (20) goes to zero. Then, the nodal approximation for the variables is written as:

\[
\delta u(x) = < N(x, \alpha) \{ \delta U_n \}; < N(x, \alpha) > = < P > [P_n]^{-1};
\]

(21)

where the nodal matrix, \( [P_n] \), and interpolation functions, \( < N(x, \alpha) >\), are

\[
[P_n] = \begin{bmatrix}
1 & 0 \\
\cos(\alpha l) & \sin(\alpha l)/\alpha \\
\end{bmatrix}; \quad < N(x, \alpha) > = \frac{\sin(\alpha l)}{\sin(\alpha l)} < \sin(\alpha l - x) \sin(\alpha x) > .
\]

Combining eqns (20) and (21), the (dynamic) stiffness matrix is obtained as:

\[
W^e = H_m[\delta u,_{x}u]^{x_{j+1}}_{x_j} = H_m[(-\delta u,_{x})_{x=0}u_1 + (\delta u,_{x})_{x=l}u_2] = < \delta u_n > [k(\alpha)]{u_n}
\]

(22)

where

\[
[k(\alpha)] = H_m \begin{bmatrix}
(-N_1,_{x})_{x=0} & (N_1,_{x})_{x=l} \\
(-N_2,_{x})_{x=0} & (N_2,_{x})_{x=l}
\end{bmatrix} = H_m \frac{\sin(\alpha l)}{\sin(\alpha l)} \begin{bmatrix}
\cos(\alpha l) & -1 \\
-1 & \cos(\alpha l)
\end{bmatrix}
\]

with \( \alpha^2 = \frac{\rho_0}{E} \) and \( l = x_{j+1} - x_j \). It can be verified that if the expression 22 is developed as a series of function of \( \alpha^2 \) [10], then

\[
[k(\alpha)] = [k(\alpha = 0)] - \alpha^2[m] - \alpha^4[m_1] \ldots
\]

(23)

The first two terms will correspond to the dynamic stiffness matrix obtained by the classical Finite Element Method (FEM) [9]

\[
[k(\omega^2)] = H_m([k] - \omega^2[m])
\]

(24)

where \([k(\alpha = 0)]\) and \([m]\) are the static stiffness and mass matrices [10]. Similar developments in series, for some vibration problems, are also presented by [9, 11].
2.3 Global discretized form for $W$

As for the classical FEM, each element integral form $W^e$ obtained from the exact DSM or the DFE approach leads to

$$W^e = \langle \delta u_n \rangle [k(\alpha)]\{u_n\}$$

where $[k(\alpha)]$ is the frequency dependent element dynamic stiffness matrix for element e. The discretized global form $W$ is the sum of the discretized elementary forms $W^e$. The standard assembly process is then used:

$$W = \sum_{\text{elements}} W^e = \langle \delta U_n \rangle [K(\alpha)]\{U_n\} = 0 \quad (25)$$

The solution of the problem consists in finding the eigenvalues $\alpha$ and vectors $U_n$ that make $W$ vanish for any arbitrary $\langle \delta U_n \rangle$ while satisfying the boundary conditions defined on $S_u$ (i.e., $u = u_e$ and $\delta u = 0$ and $\delta U_i = 0; U_i = \bar{U}_i$ for all the degrees of freedom $U_i$ having specified value of $\bar{U}_i$). The boundary conditions can be introduced in the expression (25) by well-known methods (e.g., Direct method or Penalty method).

2.4 Eigenvalue calculation (Sturm sequence property of the DSM)

Powerful algorithms exist for determining any number of natural frequencies in the case of linear eigenvalue problems, resulting from the discrete or lumped mass models. Utilizing the Sturm sequence property, this enables one to determine with ease how many natural frequencies lie below any chosen frequency, thereby making it possible to converge on any particular natural frequency, to any required accuracy. A comparable method was presented by Williams and Wittrick [4] for the type of nonlinear eigenvalue problems mentioned above, where we shall consider undamped, linearly elastic models, and in which the strain energy is a quadratic function of the displacements. The natural frequencies of an infinite system are therefore calculated by varying $\omega$ in small steps, calculating the determinant of $K_{DS}$ at each step, and by seeking the values which make the determinant zero. Consequently, the corresponding mode shapes can be evaluated [4, 9].

3 Application of the theory and concluding remarks

It can be verified that if $\omega$ (and then $\alpha$) $\to 0$, the basis functions $\langle P \rangle$ change to $\langle 1 \ x \rangle$ and the shape functions $\langle N(x, \alpha) \rangle$ change to $\langle 1 - x/l \ x/l \rangle$ which are, respectively, the basis functions and shape functions, obtained for a linear approximation, in the classical FEM. As it can be observed, the static stiffness matrix $[k]$ can be obtained from eqn (22) (or 23 by putting $\alpha = 0$). The static mass matrix $[m]$ can also be found from dynamic formulation as presented in [10, 9]. As already discussed, “exact” member (element) equations exist for important structures, including plane frames, space frames, grids, and many plate and shell problems. The frequency dependent Dynamic Stiffness Matrix (DSM) formulation leads to the exact natural frequencies and modes of vibration of structures. In spite of its very accurate results, the most important weakness of the DSM formulation is that it is not
a general formulation. The Finite Element Method (FEM), on the other hand, provides a general formulation tool where element matrices are usually evaluated from assumed fixed shape functions. As shown in the previous section, the exact frequency calculation for axial vibration of simple beams can be formulated based on the FEM approach when the appropriate Dynamic (frequency dependent) Trigonometric Shape Functions (DTSFs) are used. In other words, the FEM can be directly used to derive the DSM formulation for certain problems [9, 10] (see Appendix A and Figures 2 and 3 for analytical and graphical representations of axial, torsional and bending DTSFs). On the other hand, the DFE formulation, together with appropriate DTSFs, can be also exploited to formulate complex cases where the closed form solution of the governing differential equations are not available (e.g., centrifugally loaded pure lateral and coupled Bending-Torsion vibrations of uniform/nonuniform beams where the coefficients of the governing ODEs are not constant). Although the resulting DFE matrices, in these cases, are not exact but the method shares the generality of FEM method. Consequently, it can be easily extended to model beams with nonconstant mechanical and geometrical properties which distinguishes DFE method from DSM. Figure 4 shows the comparison between FEM and DFE methods for different frequencies of a linearly tapered beam. DFE results in much higher convergence rates, when compared to FEM.

Figure 2: The variation of the dynamic in-plane (lead-lag) flexural shape functions \(N_i\) vs frequency changes for a uniform beam rotating in horizontal plane; \(E = 1 \text{ GPa}, A=1 \text{ m}^2, L=1 \text{ m}, \rho =1 \text{ kg/m}^3, I=1 \text{ m}^4, \) and an axial force equivalent to the rotation speed \(\Omega =12 \text{ rad/sec.}\) (a) \(N_1;\) (b) \(N_2;\) (c) \(N_3;\) (d) \(N_4;\) \(-x-, \omega_1, 1^{st} \text{ natural frequency}; \)-\(\Delta -, \omega_2, 2^{nd} \text{ natural frequency}; \)-\(\star-, \omega_3, 3^{rd} \text{ natural frequency}; \)-\(\circ-, \omega_4, 4^{th} \text{ natural frequency}.)
Figure 3: The change of the fourth dynamic shape function, $N_4$, at the third natural frequency, $\omega_3$, vs spinning speed for the same beam as in Figures 2 for different distributed axial forces equivalent to: $\times -\times$, $\Omega=4 \text{ rad/s}$; $\triangle -\triangle$, $\Omega=8 \text{ rad/s}$; and $\ast -\ast$, $\Omega=12 \text{ rad/s}$.

Figure 4: FEM and DFE convergence results for the first four frequencies of lateral vibration for a cantilever linearly tapered beam (taper ratio $= 0.5$).

Acknowledgment

The author wishes to acknowledge the support provided by Ryerson University.

References


APPENDIX A: Dynamic (frequency-dependent) shape functions

The extensional dynamic shape functions are found to be as

\[ N_1(\omega)_e = \cos(\eta \xi) - \cos(\eta) \frac{\sin(\eta \xi)}{D_a} ; \quad N_2(\omega)_e = \frac{\sin(\eta \xi)}{D_a} ; \]

and torsional dynamic shape functions are

\[ N_1(\omega)_t = \cos(\tau \xi) - \cos(\tau) \frac{\sin(\tau \xi)}{D_t} ; \quad N_2(\omega)_t = \frac{\sin(\tau \xi)}{D_t} ; \]

The dynamic flexural shape functions are found to be as follows [2]

\[ N_1(\omega) = \frac{(\alpha \beta)}{D_f} \left[ -\cos(\alpha \xi) + \cos(\alpha (1 - \xi)) \cosh(\beta) + \cos(\alpha) \cosh(\beta (1 - \xi)) - \cosh(\beta \xi) - \frac{\beta}{\alpha} \sin(\alpha (1 - \xi)) \sinh(\beta) + \frac{\alpha}{\beta} \sin(\alpha) \sinh(\beta (1 - \xi)) \right] ; \]
\[ N_2(\omega) = \frac{1}{D_f} \left[ \beta \left( \cosh(\beta (1 - \xi)) \sin(\alpha) - \cosh(\beta) \sin(\alpha (1 - \xi)) - \sin(\alpha \xi) \right) + \alpha \cos(\alpha (1 - \xi)) \sinh(\beta) - \cos(\alpha) \sinh(\beta) + \sin(\alpha \xi) \sinh(\beta (1 - \xi)) - \sinh(\beta \xi) \right] ; \]
\[ N_3(\omega) = \frac{(\alpha \beta)}{D_f} \left[ -\cos(\alpha (1 - \xi)) + \cos(\alpha \xi) \cosh(\beta) - \cosh(\beta (1 - \xi)) + \cos(\alpha) \cosh(\beta) - \frac{\beta}{\alpha} \sin(\alpha \xi) \sinh(\beta) + \frac{\alpha}{\beta} \sin(\alpha) \sinh(\beta \xi) \right] ; \]
\[ N_4(\omega) = \frac{1}{D_f} \left[ \beta \left( -\cosh(\beta \xi) \sin(\alpha \xi) + \sin(\alpha (1 - \xi)) + \cosh(\beta) \sin(\alpha \xi) - \alpha \cos(\alpha) \sinh(\beta) + \sinh(\beta (1 - \xi)) + \cos(\alpha) \sinh(\beta) \right) \right] ; \]

where

\[ D_f = (\alpha \beta) \left[ -2 (1 - \cos(\alpha) \cosh(\beta)) + \frac{(\alpha^2 - \beta^2)}{\alpha^2} \sin(\alpha) \sinh(\beta) \right] ; \]
\[ D_a = \sin(\eta) ; \quad D_t = \sin(\tau) ; \quad \alpha, \beta = \frac{1}{2} \sqrt{\frac{1}{[2 \times A]^3} B \pm |B^2 - 4 A \times C|^{\frac{3}{2}}} ; \]
\[ \eta = \omega k (m/H_a)^{1/2} ; \quad \tau = \omega k R (m/H_t)^{1/2} ; \quad A = \frac{H_t}{k} ; \quad B = -\left( \frac{P}{k} \right) ; \quad C = -ml_k (\omega^2) . \]