Experimental modal analysis of a water-filled circular cylindrical tank

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Abstract

Experimental modal analysis has been performed on an empty and water-filled, circular cylindrical tank horizontally suspended. The tank is composed of a circular cylindrical shell with a longitudinal butt seam welding and two end annular plates. A rubber disk was glued to each annular plate to close the tank. The boundary conditions imposed to the circular cylindrical shell by the thin end plates can be considered very close to simply supports. Two little pipe fittings, used for supplying water, were welded to one of the end plates in such a position that they do not affect shell vibrations. The tank was hung to a truss with its axis kept horizontal by a compliant suspension, by using cables connected to the end plates. The Frequency Response Functions (FRFs) between 100 response points and one single excitation point (SIMO test) have been measured. Burst random has been used as excitation input. Natural frequencies and mode shapes are in very good agreement with theoretical data computed using the Flügge theory of shells and potential and incompressible fluid model.

1 Introduction

Experimental modal analysis is used to study dynamic response of structures in terms of natural frequencies, mode shapes and damping factors. Circular cylindrical tanks are used in many branches of engineering to contain fluids. Berry and Reissner [1] studied free vibrations of fluid-filled, simply supported circular cylindrical shells. This work was extended by Lindholm [2] to the case of mode shapes with presence of circumferential nodal lines. The influence of a partial filling on vertical shells was also experimentally investigated in reference [2]. Approximate solutions to free vibrations of partially-filled, vertical circular
shells were introduced by Lakis and Paidoussis [3], who applied a hybrid finite element method, and by Gonçalves and Batista [4]. Rigorous solutions, taking into account sloshing modes of the free surface of the fluid were given by Gonçalves and Ramos [5] and Amabili et al. [6]. Experimental results on free vibrations of partially and completely filled shells were given by Amabili and Dalpiaz [7] for simply supported shells, Chiba [8] for clamped shells, Chiba et al. [9] and Mazluch et al. [10] for clamped-free shells. Free vibrations of partially filled, horizontal shells were theoretically and experimentally studied by Amabili [11, 12].

In the present work, an experimental modal analysis has been performed on an empty and water-filled tank horizontally suspended. The tank has been built to study vibrations of a simply supported circular cylindrical shell. A grid of 100 measure points has been utilised to describe the geometry of the shell. A burst random excitation has been applied on a single point by means of an electrodynamic shaker. Comparison of theoretical and experimental natural frequencies and mode shapes has been satisfactorily done for the empty and completely water-filled shell. Water effect on natural frequencies, mode shapes and damping ratios of the shell has also been analysed.

A different excitation technique was used with respect to Amabili and Dalpiaz [7], who used an impact excitation, in order to better excite higher-frequency modes.

2 Theoretical approach

A simply supported circular cylindrical shell of length \( L \), radius \( R \) and uniform thickness \( h \) is assumed. A cylindrical co-ordinate system is introduced \((\theta;x,r,\theta)\) with the origin at one shell edge. The components of displacement of a point on the mean surface in the axial, angular and radial directions are indicated by \( u \), \( v \) and \( w \) respectively. The symmetric mode shapes of the shell, with respect to the coordinate \( \theta = 0 \), are given by [13]:

\[
\begin{align*}
\begin{bmatrix} u \\ v \\ w \end{bmatrix} = & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \sum_{j=1}^{3} \left[ A_{nmj} \cos(n\theta)\cos(m\pi x / L) \\ B_{nmj} \sin(n\theta)\sin(m\pi x / L) \\ \cos(n\theta)\sin(m\pi x / L) \right],
\end{align*}
\]

where \( A_{nmj} \) and \( B_{nmj} \) are the mode shape coefficients (the normalisation of the radial displacement to 1 gives a very high value, theoretically \( \infty \), to \( B_{nmj} \) for the axisymmetric mode \((n=0)\) with prevalent angular displacement); \( n \), \( m \) and \( j \) indicate respectively the number of circumferential waves, the number of axial half-waves and the mode number \( j = 1, 2, 3 \) which denotes modes with prevalent radial, axial and angular displacements.

To model the vibrational behaviour of the liquid-filled or in vacuo shell, the Flügge theory of shells as been used. By applying the solutions (1) into the Flügge differential operator we obtain the dynamic matrix \( D \) [11,13]:
 Fluid Structure Interaction

\[
\mathbf{D} = \begin{bmatrix}
\lambda^2 + (1+k)(1-\nu)n^2/2 & -\lambda n(1+\nu) & -\lambda\nu - k\lambda(\lambda^2 - (1-\nu)n^2/2) \\
-\lambda n(1+\nu)/2 & n^2 + (1+3k)(1-\nu)\lambda^2/2 & n + (3-\nu)kn\lambda^2/2 \\
-\lambda\nu - k\lambda(\lambda^2 - (1-\nu)n^2/2) & n + (3-\nu)kn\lambda^2/2 & (1+k(\lambda^2 + n^2)^2 + k(1-2n^2))
\end{bmatrix}
\]

(2)

where 

\[
\lambda = \frac{m\pi R}{L}, \quad k = \frac{h^2}{12 R^2}, \quad \chi = \frac{1}{\rho_s} \left( \frac{\rho_f L}{m\pi R} \right) \frac{I_{n+1}(\lambda)}{I_{n+1}(\lambda)}.
\]

(3)

\(v\) is the Poisson ratio, \(\rho_s\) and \(\rho_f\) in eqns (3) are the shell mass density and the fluid mass density respectively; \(I_n\) is the modified Bessel function of order \(n\) and \(I_{n+1}\) its derivative. For the empty shell we consider \(\rho_f = 0\) and \(\chi = 1\).

The frequencies and modes of a circular cylindrical shell can be found by solving the eigenvalue problem:

\[
\det(\mathbf{D} - \Omega^2 \mathbf{I}) = 0,
\]

(4)

where \(\Omega\) is the non-dimensional frequency parameter defined as:

\[
\Omega^2 = \frac{\omega^2 R^2 \rho_s (1-\nu^2)}{E},
\]

(5)

\(\omega\) is the corresponding circular frequency and \(E\) is the Young’s modulus. The eigenvectors of eqn (4) give mode shapes. The eigenvalue problem leads to a bicubic equation which has three roots \(\Omega_{nmj} (j = 1, 2, 3)\) for any given values of \(n\) and \(m\).

The solution of the problem is also given by anti-symmetric modes with respect to the axis \(\theta = 0\):

\[
\begin{bmatrix}
u \\ \omega
\end{bmatrix} = \sum_{n=0}^{\infty} \sum_{m=1}^{3} \begin{bmatrix} A_{nmj} \sin(n\theta) \cos(m\pi x / L) \\ B_{nmj} \cos(n\theta) \sin(m\pi x / L) \\ \sin(n\theta) \sin(m\pi x / L)
\end{bmatrix}.
\]

(6)

Therefore we have two identical mode shapes (the symmetric and the anti-symmetric one) with the same natural frequency. The two modes have the same shape but are shifted angularly of \(\pi/(2n)\).

The fluid contained in the tank has been assumed to be inviscid, irrotational and incompressible, so that the motion can be described by means of the velocity potential, which satisfies the Laplace equation (eqn (7)): 
\[ \nabla^2 \Phi(x, r, \theta, t) = 0. \]  

(7)

where \( x, r \) and \( \theta \) are defined as above; \( t \) is the time.

3 Experimental procedure

Modal tests have been performed on a tank horizontally suspended. The specimen is represented below (Figure 1):

The specimen had been obtained by welding two 0.25 mm thick annular plates to a commercial pipe having length \( L = 520 \) mm, radius \( R = 149.4 \) mm and thickness \( h = 0.519 \) mm. The end plates have an inner radius of 60 mm and an outer radius equal to the outer shell radius. All the parts are made of AISI 304 stainless steel having the following characteristics: Young’s modulus \( E = 198 \) GPa, Poisson ratio \( \nu = 0.3 \) and mass density \( \rho_s = 7800 \) kgm\(^{-3}\). The circular shell has a longitudinal butt seam welding. A rubber disk was glued to each end annular plate to close the tank and avoid overpressure. The boundary conditions imposed to the circular cylindrical shell by the thin end plates can be considered very close to simply supports. Two little pipe-fittings were welded onto one of the end plates in such a position that they do not affect shell vibrations; they are used for supplying water. The tank was hung to a truss with its axis kept horizontal by a compliant suspension, by using cables connected to the end plates. The stinger has a diameter of 1 mm and a length of 58 mm.

In this analysis we only consider the modes of prevalent radial displacements \( (j = 1) \) which are always the modes with the lowest frequency.

The Frequency Response Functions (FRFs) were measured between 100 response points and one single excitation point. Both excitation force and measured responses were in radial direction. The response points were located on a grid with 5 equidistant \( (L / 4) \) circumferences in longitudinal direction and 20
positions around the circumferences; this allows detection of modal shapes having up to 10 circumferential waves.

The LDS V406 shaker with a LDS PA100E power amplifier has been used to produce a burst-random excitation with the following parameters (Table 1):

Table 1: Burst-Random excitation parameters; water-filled and empty tank.

<table>
<thead>
<tr>
<th></th>
<th>Water-filled shell</th>
<th>Empty shell</th>
</tr>
</thead>
<tbody>
<tr>
<td>Minimum frequency</td>
<td>0 Hz</td>
<td>0 Hz</td>
</tr>
<tr>
<td>Maximum frequency</td>
<td>1198 Hz</td>
<td>2500 Hz</td>
</tr>
<tr>
<td>Burst length</td>
<td>70 %</td>
<td>55 %</td>
</tr>
<tr>
<td>Frequency resolution</td>
<td>0.146 Hz</td>
<td>0.152 Hz</td>
</tr>
<tr>
<td>No. of sample points</td>
<td>8192</td>
<td>16384</td>
</tr>
<tr>
<td>No. of averages</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>Window type</td>
<td>Uniform</td>
<td>Uniform</td>
</tr>
</tbody>
</table>

The input force has been measured using a B&K 8200 piezoelectric force transducer, weighting 21 grams. Figure 1 shows the load cell positioned near the shaker to avoid an added mass effect and the two accelerometers B&K 4393, weighting 2.4 grams, used to measure the response acceleration. The FRF of the driving point and its coherence are reported in Figure 2.

![Figure 2: FRF of the driving point and its coherence](image)

The acquisitions and the modal parameter estimations have been conducted on a HP C3000 Workstation with DIFA Scadas II acquisition front-end and LMS CADA-X 3.5.B software. The FRFs have been estimated using the $H_v$ (noise in Input & Output) technique. The modal parameters have been estimated using the Time Domain MDOF ASM method for the water-filled shell and the Frequency Domain MDOF ASM for the empty shell. To determine frequency and damping
the Time Domain MDOF ASM uses the Least Square Complex Exponential (LSCE) parameter estimation technique while the Frequency Domain MDOF ASM uses the Frequency domain Direct Parameter Identification (FDPI) technique.

The model has been validated using the Modal Assurance Criterion and the Modal Phase Collinearity (MPC). The validation of the modal model has been done by using MAC, MPC and MOV (Modal overcomplexity mode) indexes. The MAC value between corresponding modes should be near to 100% while between two linearly independent modes is small, approximately zero, therefore the MAC matrix (reported in Table 2 only for the modes with $m = 1$, $\zeta$ is the coefficient of damping %) has been used to check the orthogonality of modes. MPC index expresses the linear functional relationship between the real and the imaginary parts of the unscaled mode shape vector. The index should be high (near 100 %) for real normal modes. The MOV index should be high (near 100 %) for high quality modes.

Double poles with orthogonal eigenvectors, due to the axial-symmetry of the structure have been detected. Theoretically double-poles are associated to the same frequency; however, due to the small imperfections of the shell the two frequencies are not coincident but very close.

<table>
<thead>
<tr>
<th>Natural Frequency (Hz)</th>
<th>75.01</th>
<th>78.39</th>
<th>85.35</th>
<th>86.08</th>
<th>87.27</th>
<th>110.26</th>
<th>111.14</th>
<th>117.47</th>
<th>117.96</th>
<th>149.68</th>
<th>147.75</th>
<th>162.93</th>
<th>247.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAC</td>
<td>0.10%</td>
<td>0.09%</td>
<td>0.11%</td>
<td>0.11%</td>
<td>0.15%</td>
<td>0.16%</td>
<td>0.21%</td>
<td>0.15%</td>
<td>0.33%</td>
<td>0.33%</td>
<td>0.15%</td>
<td>0.19%</td>
<td>0.12%</td>
</tr>
<tr>
<td>MPC</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>MOV</td>
<td>96.3</td>
<td>99.7</td>
<td>98.4</td>
<td>99.4</td>
<td>99.4</td>
<td>99.2</td>
<td>99.1</td>
<td>94.4</td>
<td>96.9</td>
<td>98.5</td>
<td>98.1</td>
<td>98.2</td>
<td>99.2</td>
</tr>
</tbody>
</table>

4 Results and discussion

Experimental results concerning the measured modal parameters are presented and compared to theoretical calculations both for the completely filled and the empty shell. Theoretical and experimental values are represented in Figure 3 for the modes with $m = 1, 2, 3$ and $n = 3 \ldots 10$. Theoretical results have been computed finding the solution of the eigenproblem (4), considering only modes with prevalent radial displacements which are, in the present case, the lower frequency modes.

Theoretical and experimental results are in good agreement both in natural frequencies and mode shapes.
Figure 3: Theoretical and experimental natural frequencies for the water-filled (a) and empty shell (b): Theoretical results: -, m=1; ..., m=2; --., m=3. Experimental results: □, m=1; x, m=1 2nd mode; Δ, m=2; +, m=2 2nd mode; ○, m=3.

In Figure 4 and 5 eighth vibration modes with m=1 are ordered with the number of circumferential waves \( n \) for the water-filled and the empty shell respectively. The results of experimental modal analysis show that couples of orthogonal modes with the same mode shape, due to the symmetry of the shell, are present. The dashed line represents the 2nd mode when a couple of modes has double-poles have been detected. Also mode shapes are in very good agreement with the theory.
Figure 4: Experimental modes; o• measured points; -,- --- interpolating lines from Fourier analysis; + driving point; water-filled tank.

Figure 5: Experimental modes; o• measured points; -,- --- interpolating lines from Fourier analysis; + driving point; empty tank.
The parameter estimation for the empty shell modal model kept to some problems because of the added mass effect of the two accelerometers. In fact the empty shell is a very thin and light structure and the presence of the two accelerometers gave place to a leftwards shift in frequency in the measured FRFs, different for each measurement point. The parameter estimation methods work in the FRFs' summation and a lot of broad and jagged FRF peaks were found; this brought to difficulties in estimating modal parameters (see Figure 6). This is the reason why a different parameter estimation method was used for the parameter estimation of the empty shell modal model. The Frequency Domain MDOF ASM better suited the parameter estimation for the empty shell modal model.

![Leftwards shifts in frequency due to the added mass effect of the two accelerometers](image)

Figure 6: FRF sum blocks (added mass effect of the two accelerometers gives a leftwards shift in frequency)

In conclusion, the effect of the fluid in a simply supported tank with a glued rubber disk to the end annular plates is a large reduction of the natural frequencies of the shell because of the added mass of the fluid.

**Acknowledgements**

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References


