Flutter analysis of suspension bridges

J.R. Banerjee  
Department of Aeronautical, Civil and Mechanical Engineering,  
City University, Northampton Square,  
London EC1V 0HB, England.

Abstract

A procedure for flutter analysis of suspension bridges using the normal mode method and two-dimensional unsteady aerodynamics is presented in this paper. The dynamic stiffness method using bending-torsion coupled beam theory with warping stiffness included is used in the modal analysis whereas the aerodynamic forces are modelled using the Theodorsen type flat plate theory. The generalized mass, stiffness and aerodynamic matrices are obtained in modal coordinates. The flutter matrix is then formulated by summing algebraically the generalized mass, stiffness and aerodynamic matrices. Finally the flutter determinant is solved for flutter speed and flutter frequency. This is achieved by locating the flutter condition when both the real and imaginary parts of the flutter determinant (hence the whole flutter determinant) are zero. The method is demonstrated by numerical results obtained for the Innoshima bridge situated between Honshu and Shikoku in Japan.

1 Introduction

There are several papers [1-10] in the published literature dealing with the flutter analysis of suspension bridges. For instance Agar [1] and Miyata and Yamada [2] sought solution for the flutter problem of suspension bridges using numerical methods based on normal modes and Theodorsen type unsteady aerodynamics. Scanlan and Jones [3] presented an empirically based formulation for flutter analysis in the frequency domain using flutter derivatives. Their investigation is centred on experimentally determined flutter derivatives,
and a full three-dimensional modal analysis of the structure. Tinh [4] gave a theoretical formula to determine the critical flutter speed of a suspension bridge and tabulated his results for a wide range of parameters so that estimates of flutter speed for different suspension bridges can be obtained. Agar [5] extended his earlier work [1] and addressed the issue of how the degree of refinement of the basic structural model and the number of natural modes included in the analysis affect the flutter prediction. Kobayashi and Nagaoka [6] showed the effect of an active control technology in suppressing the flutter of suspension bridges. The identification of flutter derivatives by experimental means can be found in the work of Zasso, Cigada and Negri [7] and Singh, Jones, Scanlan and Lorendeaux [8]. Beith [9] used a practical engineering method by de-coupling the equation of motion to response in each mode while investigating the flutter characteristics of long span bridges. Katsuchi, Jones and Scanlan [10] presented an analytical investigation on multimode coupled flutter and buffeting of the Akashi-Kaikyo bridge and compared their results with related wind tunnel tests. Their theoretical results compared favourably with experimental ones.

However, the object of this paper is to use the normal mode method in conjunction with generalized coordinates to solve the flutter problem of suspension bridges. In the structural idealization it is assumed that the overall bridge structure can be modelled by using dynamic stiffness elements of bending-torsion coupled beams with warping included. Theodorsen's theory [11] is used in the aerodynamic idealisation and the unsteady aerodynamic forces are expressed in modal coordinates. From a selected number of normal modes the generalised mass, stiffness and aerodynamic matrices are obtained and then they are summed algebraically to construct the resulting flutter matrix. The determinant of this matrix, namely the flutter determinant, is a complex function of both air speed and frequency. The zeros of the determinant, which give the flutter speed and flutter frequency, are obtained by evaluating its real and imaginary parts over a wide range of air speed and frequency. For flutter condition to occur, both the real and imaginary parts of the flutter determinant must be zero, yielding the flutter speed and flutter frequency.

2 Theory

The co-ordinate system and notation for an idealised suspension bridge are shown in Figure 1. The bridge is considered to be a bending-torsion coupled beam, possessing representative, but effective, values of bending \((EI)\), torsional \((GJ)\) and warping \((ET)\) rigidities, mass per unit length \((m)\) and mass moment of inertia per unit length \((I_o)\) respectively. The free vibration analysis of a structure consisting of such beams has been carried out by Banerjee, Guo and Howson [12] by using the dynamic stiffness matrix method. The same procedure is applied here to study the free vibration characteristics of suspension bridges, which are fundamental prerequisites before carrying out a flutter analysis. However, attention here is confined to the analysis of a suspension bridge with uniform properties along its length, although the theory put forward by
Banerjee, Guo and Howson [12] can account for non-uniform distribution of properties by considering the suspension bridge as an assemblage of many different bending-torsion coupled beams. The length and width (semi-width) of the bridge are taken to be \( l \) and \( 2b \) \((b) \) respectively. The elastic axis is assumed to be at a distance \( ba_h \) from the mid-chord (mid-width) position whereas the mass axis is assumed to be at a distance \( bx_\alpha \) from the elastic axis as shown. (Note that \( a_h \) and \( x_\alpha \) are both non-dimensional quantities expressed as fractions of semi-chord and they are positive in the positive direction of \( X \) as shown.) The elastic axis, which is coincident with the \( Y \)-axis, is allowed to deflect out of plane by \( h(y, t) \), while the cross-section is allowed to rotate or twist about \( OY \) by \( \psi(y, t) \), where \( y \) and \( t \) denote distance from the origin and time respectively.

### 2.1 Free vibration analysis

Using bending-torsion coupled beam theory with warping stiffness included, the governing differential equations of motion in free vibration of a suspension bridge can be written as [12]

\[
EI \frac{\partial^4 h}{\partial y^4} + m \frac{\partial^2 h}{\partial t^2} - mbx_\alpha \frac{\partial^2 \psi}{\partial t^2} = 0
\]

\[
EI \frac{\partial^4 \psi}{\partial y^4} - GJ \frac{\partial^2 \psi}{\partial y^2} + I_\alpha \frac{\partial^2 \psi}{\partial t^2} - mbx_\alpha \frac{\partial^2 h}{\partial t^2} = 0
\]

Assuming harmonic oscillation with circular frequency \( \omega \), then

\[
h(y, t) = H(y)e^{i\omega t}
\]

\[
\psi(y, t) = \Psi(y)e^{i\omega t}
\]

Substituting eqn (3) into eqns (1) and (2) gives

\[
EI \frac{d^4 H}{dy^4} - m\omega^2 H + mbx_\alpha \omega^2 \Psi = 0
\]

and

\[
EI \frac{d^4 \Psi}{dy^4} - GJ \frac{d^2 \Psi}{dy^2} - I_\alpha \omega^2 \Psi + m\omega^2 bx_\alpha H = 0
\]

Using the above equations, the dynamic stiffness matrix of the bending-torsion coupled beam, which relates harmonically varying forces with harmonically varying amplitudes at the ends of the beam, can be developed using the theory described in Ref. [12]. The resulting dynamic stiffness matrix can then be used to perform a modal analysis of a bridge structure yielding natural frequencies and mode shapes. Once this is accomplished, one can proceed to perform flutter analysis using the normal mode method and linking the elastic and inertia forces to unsteady aerodynamics forces through the use of generalised coordinates.
2.2 Generalised mass and stiffness matrices

The mass and stiffness matrices of a suspension bridge can now be reduced to diagonal form to give generalised mass and stiffness matrices. This is achieved by using the normal (orthogonal) modes obtained from the free vibration analysis. The procedure is briefly summarised as follows.

If \([\Phi]\) is the modal matrix formed by the selection of normal modes so that each column of \([\Phi]\) represents a normal mode \(\Phi_i\), then the generalised mass and stiffness matrices \([M_G]\) and \([K_G]\) can be obtained by post multiplying the mass and stiffness matrices \([M]\) and \([K]\) by the modal matrix \([\Phi]\) and then pre-multiplying the resultant matrices by the transpose of the modal matrix, i.e. \([\Phi]^T\).

In matrix notation

\[
\]

Clearly, \([M_G]\) and \([K_G]\) are diagonal matrices and if the number of modes chosen is \(n\), the order of these matrices will be \(n\times n\).

2.3 Generalised aerodynamic coefficients

The generalized aerodynamic coefficients can be derived by applying the principle of virtual work. The aerodynamic strip theory of Theodorsen for unsteady lift and moment [11, 13-15] and the normal modes obtained above, can be used when applying the principle of virtual work. Thus if the bending displacement and torsional rotation in the \(i\)-th mode are \(H_i(y)\) and \(\Psi_i(y)\), the elements of the generalized aerodynamic matrix \([QA]\) can be expressed as [16]

\[
QA_{ij} = \int_0^l \left( A_{11} H_i H_j + A_{12} H_i \Psi_j + A_{21} H_i \Psi_j + A_{22} \Psi_i \Psi_j \right) dy 
\]

In eqn (8) \(A_{11}, A_{12}, A_{21}, A_{22}\) are related to the unsteady lift and moment \(L\) and \(M\) as follows [13-15]

\[
\begin{bmatrix}
L(y) \\
M(y)
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
h(y, t) \\
\psi(y, t)
\end{bmatrix}
\]

with
Fluid Structure Interaction

\begin{align}
A_{11} &= -\pi \rho U^2 \{ -k^2 + 2C(k)ik \} \\
A_{12} &= \pi \rho U^2 b [(a_h k^2 + ik) + 2C(k)\{1 + ik(0.5 - a_h)\}] \\
A_{21} &= -\pi \rho U^2 b\{2C(k)ik(0.5 + a_h) - k^2 a_h\} \\
A_{22} &= \pi \rho U^2 b^2 [2(0.5 + a_h)C(k)\{1 + ik(0.5 - a_h)\} + \frac{k^2}{8} + k^2 a_h^2 + (a_h - 0.5)ik]
\end{align}

In eqns (10)-(13), \( U, b, \rho, k, C(k) \) and \( a_h \) are in the usual notation: the airspeed, semi-chord, density of air, reduced frequency parameter (defined as \( k = \omega b / U \)), Theodorsen function and elastic axis location from mid-chord respectively \([13-15]\). Note that the signs of \( A_{11} \) and \( A_{21} \) as given in Ref. \([14]\) have been reversed because \( h \) is considered positive upward in this paper.

2.4 Formulation of the flutter problem and solution

The flutter matrix is obtained by summing algebraically the generalized mass, stiffness and aerodynamic matrices. Thus for a system without structural damping the flutter matrix \([QF]\) can be formed as given by eqn (14) below. (Structural damping has generally a small effect on the oscillatory motion and is not included here.)

\[
[QF]\{q\} = [-\omega^2[M_G] + [K_G] - [QA]]\{q\}
\]

where \([QA]\) is the complex \( n \times n \) generalized aerodynamic matrix whose elements are defined in eqn (8), \([M_G]\) and \([K_G]\) are \( n \times n \) diagonal matrices of generalized mass and generalized stiffness respectively (with the \( i \)-th diagonal representing the generalized mass \( M_i \) and generalized stiffness \( K_i \)), \{\( q \}\) is the column vector of \( n \) generalized co-ordinates (which are functions of time) and \( \omega \) is the circular frequency in rad/s.

For flutter to occur, the determinant of the complex flutter matrix must be zero so that from eqn (14)

\[
|QF| = |{-\omega^2[M_G] + [K_G] - [QA]}| = 0
\]

The solution of the flutter determinant is a complex eigen-value problem because the determinant is primarily a complex function of two unknown variables, the airspeed \((U)\) and the frequency \((\omega)\). The method used selects an air speed and evaluates the real and imaginary parts of the flutter determinant for a range of frequencies. The process is repeated for a range of airspeeds until both the real and imaginary parts of the flutter determinant vanish completely.
3 Numerical results and discussion

To demonstrate the application of the theory a numerical example is chosen which is that of the Innoshima suspension bridge [17, 18] located between Honshu and Shikoku in Japan. The data used in the analysis were adapted from Ref. [17, 18] and are given as follows:

\[ EI = 8.129 \times 10^{13} \text{Nm}^2, \quad GJ = 6.457 \times 10^{11} \text{Nm}^2, \quad ET = 9.695 \times 10^{15} \text{Nm}^4, \]
\[ m = 20667 \text{ kg/m}, \quad I_\alpha = 2.136 \times 10^6 \text{ kgm}, \quad I_\alpha = 2.136 \times 10^6 \text{ kgm}, \quad x_\alpha = 0.0, \]
\[ a_h = 0.0, \quad b = 13 \text{ m}, \quad L = 770 \text{ m}. \]

Using the above data the fundamental bending and torsional natural frequencies of the bridge with simple support end conditions were computed using the dynamic stiffness method of Banerjee, Guo and Howson [12]. These are respectively \( \omega_b = 1.044 \text{ rad/s} \) and \( \omega_t = 2.5079 \text{ rad/s} \). The corresponding mode shapes for these frequencies are half sine waves because of the simple support end conditions of the bridge and they are not shown here for brevity. Next the flutter speed and flutter frequency of the bridge were computed. These are 111 m/s and 1.89 rad/s respectively. Based on these results it can be said that the calculated flutter speed of the Innoshima bridge at 111 m/s (248 mph) is reasonable and safe.

4 Conclusions

A method based on normal modes and generalised coordinates, is presented to carry out the flutter analysis of suspension bridges. Using the proposed method, the flutter speed and flutter frequency of the Innoshima suspension bridge situated in Japan have been established. The investigation has shown that the computed flutter speed is quite high and therefore, the bridge is reasonably safe and free from flutter instability.

References


Figure 1: Coordinate system and notation for an idealised suspension bridge.