Flow-induced vibrations of axially moving web subjected to shear fluid flow

M. Watanabe & N. Kobayashi
Department of Mechanical Engineering, Aoyama-Gakuin University, Japan

Abstract

This paper deals with a theoretical stability analysis of a flexible web axially moving in a fluid-filled narrow space with shear fluid flow. In the stability analysis, governing equation of motion of the axially moving web coupled with the shear fluid flow is derived from Bernoulli-Euler’s beam equation and Navier-Stokes equations, and characteristic equation of the system is derived as a function of the axially moving speed of the web. The stability of the system is examined by calculating roots of the characteristic equation of the system. As a result, the analytical results find that divergence type instability and flutter type instability occur in the axially moving web due to the shear fluid flow when the web speed becomes high. Moreover, the analytical results show the instability regions of the divergence and flutter.

1 Introduction

Axially moving webs are seen in many industrial fields such as a copy machine, rolling machine and so forth. In these types of machines, the web is moved in a narrow space filled with fluid, and the moving web is subjected to shear fluid flow generated around the web. In particular, the fluid flow has a significant effect on the stability of the moving web when the moving speed becomes high. Moreover, in the high-speed region, flow-induced vibration and buckling phenomena occur in the axially moving web due to the shear fluid flow.

Up to present time, many research works [1]-[5] have been conducted for vibrations of an axially moving belt and string without the effect of the fluid flow. In these studies, natural frequencies and dynamic stability of the axially moving belt and string was examined.

Some research works [6]-[11] have been conducted for the stability of a
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plate subjected to high-speed fluid flow. In these studies, it was reported that flutter instability and divergence instability occur in the plate due to the fluid flow. However the stability of the axially moving web subjected to the shear fluid flow is not yet clarified sufficiently.

In this paper, we examine the stability of the web axially moving in a fluid-filled narrow space with shear fluid flow and the instability regions for various parameters.

2 Dynamic stability analysis

2.1 Modeling and coordinate system

Figure 1 shows an analytical model of a web axially moving in a fluid-filled narrow space together with a stationary coordinate system $x-y$ and geometrical parameters considered in this model. The web supported by two rollers is moved in the middle of the finite narrow space with speed $V$ and a tensile force $S_t$, and the web is subjected to shear fluid flow caused by the axially moving motion of the web. In this model, the length of the narrow space is $L$, and the gap width between the web and the side wall is $H_0$. The thickness of the web is $h$.

![Analytical model of axially moving web with shear fluid flow in fluid-filled narrow space](image)

2.2 Basic equation of axially moving web

The equation of motion of the axially moving web subjected to shear fluid flow is derived in terms of transverse displacement $w$ and with respect to the stationary coordinate system $x-y$ as follows:

$$\rho h \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 w + C_s \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) w - S_t \frac{\partial^2 w}{\partial x^2}$$

$$+ \frac{Eh^3}{12} \frac{\partial^4 w}{\partial x^4} + \frac{E^* h^3}{12} \frac{\partial^4}{\partial t^4} \left( \frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right) w = F(x,t)$$

where $t$ is time. $E$ and $E^*$ are Young's modulus and the coefficient of Kelvin-Voigt type internal dissipation, respectively. $C_s$ is the coefficient of viscous damping. $\rho$ is density of the web. $F$ is fluid force acting on the web surface per unit area.
2.3 Basic Equation of Fluid Flow

Integrating the continuity and momentum equations of fluid motion with respect to $y$ over the gap width $H(x,t)$ varying with time, the equations of motion of the shear fluid flow in the lower space of the web (as shown in Figure 1) are derived as follows:

\[
\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}((Hu) = 0
\]

\[
\frac{\partial}{\partial t}(Hu)+\frac{\partial}{\partial x}(Hu^2) + \frac{12\mu}{\rho_f H} (u-U_0) - \frac{1}{2}((V-u)^2 f_b - u^2 f_s) + \frac{H}{\rho_f} \frac{\partial P}{\partial x} = 0
\]

where $u$ is mean flow velocity over the gap, and $U_0$ is steady flow velocity. $\rho_f$, $\mu$, and $P$ are fluid density, fluid viscosity, and fluid pressure. $f_b$ and $f_s$ are fluid friction coefficients on the web surface and side wall, respectively. Here, $f_b$ and $f_s$ are given by as follows [12]:

\[
f_b = f_s = \frac{1}{4} \left(1.14 - 2 \log_{10} \left(\frac{\varepsilon_w}{D_H} + \frac{21.25}{Re^{0.9}}\right)\right)^2,
\]

where $\varepsilon_w$ and $\nu_f = (\mu / \rho_f)$ denote the wall roughness and kinematic viscosity of fluid, respectively.

Boundary conditions at the inlet ($x=0$) and the outlet ($x=L$) of the narrow space are written as follows:

\[
P(0,t) = P_{in} - \xi_{in} \frac{\rho_f}{2} u^2(0,t), \quad P(L,t) = P_{ex} + \xi_{ex} \frac{\rho_f}{2} u^2(L,t)
\]

where $P_{in}$ and $P_{ex}$ are constant fluid pressure at the inlet and outlet, $\xi_{in}$ and $\xi_{ex}$ are pressure loss coefficients at the inlet and outlet, respectively. In this paper, we use $\xi_{in} = \xi_{ex} = 0.5$ for analytical calculations.

2.4 Equations of Fluid-Structure Interaction System

In this section, we derive a governing equation of the axially moving web coupled with the fluid flow. As shown in Figure 2, the gap width $H(x,t)$ varying with time is written as follows:

\[
H(x,t) = H_0 + w(x,t)
\]

Here, the transverse displacement $w$ of the web is expressed as follows:

\[
w(x,t) = \sum_{n=1}^{M} \phi_n(x) q_n(t), \quad \phi_n(x) = \sin \lambda_n x, \quad \lambda_n = n\pi
\]

where $\phi_n(x)$ is $n$th modal function of a both-end supported beam and $q_n(t)$ is a modal coordinate.

Figure 2: Axially moving web with transverse vibration
Similarly, the fluid flow velocity $u$ and fluid pressure $P$ are expressed as follows:

$$u(x, t) = U_0 + \Delta u(x, t), \quad P(x, t) = P_0 + p(x, t)$$  \hspace{1cm} (7)

where, $U_0$ and $P_0$ are steady flow velocity and steady fluid pressure, respectively. $\Delta u$ and $p$ are unsteady flow velocity and unsteady fluid pressure. Here, $\Delta u$ and $p$ are induced by the transverse motion of the web.

Substituting equations (5) and (7) into equations (2) and (3), the equations of fluid flow are linearized with respect to the unsteady terms as follows:

$$\frac{\partial \hat{\Delta u}}{\partial t} + \hat{H}_0 \frac{\partial \Delta u}{\partial X} + \hat{U}_0 \frac{\partial \hat{\Delta u}}{\partial X} = 0$$  \hspace{1cm} (8)

$$\hat{H}_0 \frac{\partial \Delta u}{\partial t} + \hat{U}_0 \frac{\partial \hat{\Delta u}}{\partial t} + 2 \hat{H}_0 \hat{U}_0 \frac{\partial \Delta u}{\partial X} + \hat{U}_0^2 \frac{\partial \hat{\Delta u}}{\partial X} + \gamma \Delta u + \alpha \hat{w} + \frac{\hat{h}\hat{H}_0}{\hat{\rho}} \frac{\partial \hat{P}}{\partial X} = 0$$  \hspace{1cm} (9)

where

$$\alpha = 0.5\{f_0(\hat{V} - \hat{U}_0)^2 - f_s \hat{U}_0^2\}, \quad \gamma = 12\mu\hat{h} / \hat{\rho}\hat{H}_0 + \{f_s \hat{U}_0 + f_b(\hat{V} - \hat{U}_0)\}$$  \hspace{1cm} (10)

Equations (8) and (9) are expressed by dimensionless parameters by defining the following quantities.

$$T_0 = \sqrt{\rho h L^4 / D}, \quad V_0 = L / T_0, \quad T = t / T_0, \quad \hat{V} = V / V_0, \quad \hat{U}_0 = U_0 / V_0, \quad \hat{w} = w / L,$$

$$X = x / L, \quad \hat{h} = h / L, \quad \hat{H} = H / L, \quad \hat{H}_0 = H_0 / L, \quad \Delta \hat{u} = \Delta u / V_0, \quad \hat{\rho} = \rho_0 / \rho, \quad \hat{S}_t = S_t L^2 / D,$$

$$\hat{\mu} = \mu \sqrt{L^2 / \rho D}, \quad \hat{C}_s = C_s \sqrt{L^4 / \rho D}, \quad \epsilon = E^* / ET_0, \quad \hat{p} = pL^3 / D, \quad \hat{F} = FL^3 / D$$

where $D$ denotes the flexural rigidity defined by $D = Eh^3 / 12$.

Similarly, the dimensionless equation of the axially moving web is obtained from equation (1) using the dimensionless parameters as follows:

$$\left(\frac{\partial}{\partial T} + \hat{V} \frac{\partial}{\partial X}\right)^2 \hat{w} + \hat{C}_s \left(\frac{\partial}{\partial T} + \hat{V} \frac{\partial}{\partial X}\right) \hat{w} - \hat{S}_t \frac{\partial^2 \hat{w}}{\partial X^2} + \frac{\partial^4 \hat{w}}{\partial X^4}$$

$$+ \epsilon \frac{\partial^4 \hat{w}}{\partial X^4} \left(\frac{\partial}{\partial T} + \hat{V} \frac{\partial}{\partial X}\right) \hat{w} = \hat{F}(X, T)$$  \hspace{1cm} (11)

where $\hat{F}$ denotes the dimensionless fluid force obtained from the dimensionless unsteady fluid pressure $\hat{p}$ using the relationship $\hat{F} = 2 \hat{p}$, considering the lower and upper spaces of the web.

From linearized equations (8), (9) and (11), a governing equation of the axially moving web coupled with shear fluid flow is derived as follows:

$$\sum_{n=1}^{M} \left[ s^2 \delta_{mn} + s(2 \tilde{V} a_{mn} + \delta_{mn}(\tilde{C}_s + \epsilon \lambda_s^4)) + (\tilde{V}^2 - \tilde{S}_t) b_{mn} + \tilde{C}_s \tilde{V} a_{mn} + \epsilon \tilde{V} d_{mn} + \lambda_s^4 \delta_{mn}\right] q_n(s)$$

$$= -\beta \sum_{n=1}^{M} \left[ s^2 c_{mn} + s \tilde{U}_0 (c_{mn} + \frac{\tilde{c}_{mn}}{1 + sT_d / \tilde{U}_0}) + \tilde{U}_0^2 (k_{mn} + \frac{\tilde{k}_{mn}}{1 + sT_d / \tilde{U}_0})\right] q_n(s)$$  \hspace{1cm} (12)

where $s$ is the Laplace transform operator, and $q_n(s)$ is the Laplace transform of $q_n(t)$. $\delta_{mn}$ is the Kronecker delta. Here, $\beta$ denotes the added mass parameter, and $T_d$ denotes the dimensionless time constant. $\beta$ and $T_d$ are given as follows:
From equation (12), it is found that the stability of the system is governed by the dimensionless parameters $\beta$, $\dot{\Delta}$, $\ddot{\Delta}$, $c$, $e$ and $\mu$. In particular, the added mass parameter $\beta$ that denotes the magnitude of effect of the fluid flow is a very important dimensionless parameter.

The coefficients $a_{nn}$, $b_{nn}$, $d_{nn}$, $m_{nn}$, $c_{nn}$, $k_{nn}$, $\bar{c}_{nn}$ and $\bar{k}_{nn}$ are given as follows:

$$
a_{nn} = \int \frac{\partial \phi_n}{\partial X} \phi_m dX / \int \phi_m^2 dX, \quad b_{nn} = \int \frac{\partial^2 \phi_n}{\partial X^2} \phi_m dX / \int \phi_m^2 dX, 
$$

$$
d_{nn} = \int \frac{\partial^2 \phi_n}{\partial X^2} \phi_m dX / \int \phi_m^2 dX, \quad m_{nn} = \int m_n \phi_m dX / \int \phi_m^2 dX, 
$$

$$
c_{nn} = \int c_n \phi_m dX / \int \phi_m^2 dX, \quad k_{nn} = \int k_n \phi_m dX / \int \phi_m^2 dX, 
$$

$$
\bar{c}_{nn} = \int \bar{c}_n \phi_m dX / \int \phi_m^2 dX, \quad \bar{k}_{nn} = \int \bar{k}_n \phi_m dX / \int \phi_m^2 dX 
$$

where

$$m_n(X) = -\psi_n(X) + X\psi_n(1),$$

$$c_n(X) = XA_n - \frac{\gamma}{U_0 H_0} \psi_n(X) - 2\chi_n(X) + \phi_n(0)X + \left(\frac{\gamma}{U_0 H_0} X + \xi_{im}\right)\psi_n(1),$$

$$k_n(X) = XB_n - \left(\frac{\gamma}{U_0 H_0} - \frac{\alpha}{\dot{U}_0}\right)\chi_n(X) - \phi_n(X) + \left(1 + \frac{\gamma}{U_0 H_0} X\right)\phi_n(0),$$

$$\bar{c}_n(X) = T_d \left(\frac{\gamma}{U_0 H_0} X + \xi_{im}\right) - X \right)A_n, \quad \bar{k}_n(X) = T_d \left(\frac{\gamma}{U_0 H_0} X + \xi_{im}\right) - X \right)B_n,$$

$$\chi_n(X) = \int X \phi_n(X) dX, \quad \psi_n(X) = \int X \chi_n(X) dX,$$

$$A_n = (\xi_{ex} + 2)\chi_n(1) - \phi_n(0) - (\xi_{im} + \xi_{es})\psi_n(1),$$

$$B_n = (\phi_n(1) - \phi_n(0))(1 + \xi_{ex}) + \frac{\gamma}{U_0 H_0} (\chi_n(1) - \phi_n(0)) - \frac{\alpha}{\dot{U}_0^2} \chi_n(1).$$

Moreover, in the equation (12), The steady flow velocity is given by

$$\dot{U}_0 = K_p \dot{V} \quad \text{where} \quad K_p \quad \text{is about} \quad 0.5.$$

### 2.5 Characteristic Equation of System

From equation (12), the equation of the system is rewritten in matrix form as follows:

$$[A]\{q\} = \{0\}, \quad \{q\} = \{q_1(s), q_2(s), \ldots, q_M(s)\}^T$$

where $[A]$ and $\{q\}$ denotes a $(M \times M)$ system matrix and a modal coordinate vector, respectively. The characteristic equation of the system is given by

$$\det[A] = 0.$$
dimensionless modal frequency of the web. \( \text{Re}[s] \), real part of the root \( s \),
denotes the growth rate of the displacement of the web. If \( \text{Re}[s] \) is positive, the
system loses stability. In particular, if \( \text{Re}[s] > 0 \) and \( \text{Im}[s] \neq 0 \), a flutter type
instability occurs in the system. If \( \text{Re}[s] > 0 \) and \( \text{Im}[s] = 0 \), a divergence type
instability occurs in the system. In the calculation of the equation (15), we
truncated the summation of modal number at \( M = 6 \).

3 Analytical Results

3.1 Variation of Modal Frequency and Growth Rate
As an example, Figure 3 and Figure 4 show the variation of modal frequency
\( \text{Im}[s] \) and growth rate \( \text{Re}[s] \) with increasing the web speed \( \dot{V} \), respectively.
Here, in these analytical calculations, we used the parameters of the system as
follows;
\[
L = 2m, h = 1mm, H_0 = 10mm, \varepsilon_w = 0.001mm, E = 3GPa, \rho = 1.4 \times 10^3 \text{kg/m}^3, \rho_f = 1.1 \text{kg/m}^3, \\
\mu = 18.2 \times 10^{-6} \text{Ns/m}^2, S_i = 10N/m, C_s = 0, E' = 0.
\]
In this case, the dimensionless parameters are \( \beta = 62.9, \hat{S}_i = 160, \hat{C}_s = 0, \varepsilon = 0, \mu = 6.15 \times 10^{-3} \).

In Figure 3 and Figure 4, as the web speed increases, the modal frequencies
decrease and the first-mode frequency vanishes altogether, indicating the onset of
divergence. Beyond this point, the divergence of the first-mode occurs since the
growth rate of the first-mode becomes positive. If the web speed increased further,
it is seen that the coupled-mode flutter occurs when the first-mode and the
second-mode frequencies become approximately same value.

3.2 Instability Regions
Figure 5 shows the variation of the instability region with increasing the added
mass parameter \( \beta \) for \( \hat{S}_i = 100, \hat{C}_s = 0, \varepsilon = 0, \mu = 1.0 \times 10^{-5} \).
In this figure, it is seen that as the added mass parameter \( \beta \) increases, the critical speed at which the
divergence and the flutter occur becomes lower.

Figure 6 shows the variation of the instability region with increasing the
dimensionless tensile force \( \hat{S}_i \) for \( \beta = 100, \hat{C}_s = 0, \varepsilon = 0, \mu = 1.0 \times 10^{-5} \).
In this figure, it is seen that as the tensile force \( \hat{S}_i \) increases, the critical speed becomes higher.
From this result, we can find that the increment of the tensile force stabilizes the
system.

The effects of the damping on the instability regions are shown in Figure 7
and Figure 8. Figure 7 shows the variation of the instability region with increasing the viscous damping \( \hat{C}_s \), for \( \beta = 100, \hat{S}_i = 100, \varepsilon = 0, \mu = 1.0 \times 10^{-5} \).
In this figure, it is seen that the divergence and flutter instability regions become narrow
and disappear with increasing the viscous damping. In particular, the increment of the
viscous damping increases the critical speed at which the flutter occurs and
stabilizes the system.

Figure 8 shows the effect of the viscous damping \( \hat{C}_s \) on the instability
regions for the various added mass parameter \( \beta \). In this Figure, it is seen that the increment of the viscous damping stabilizes the system and has a great effect on
the system that has a smaller added mass parameter \( \beta \). On the other hand, for the
system that has a greater added mass parameter $\beta$, the viscous damping has little effect on the stability of the system.

Figure 3: Variation of dimensionless angular frequency $\text{Im}[s]$ with increasing dimensionless web speed $\hat{V}$

Figure 4: Variation of growth rate $\text{Re}[s]$ with increasing dimensionless web speed $\hat{V}$
Figure 5: Instability regions with varying added mass parameter $\beta$ ($\hat{S}_t = 100, \hat{C}_s = 0, \varepsilon = 0, \hat{\mu} = 1.0 \times 10^{-5}$)

Figure 6: Instability regions with varying dimensionless tensile force $\hat{S}_t$ ($\beta = 100, \hat{C}_s = 0, \varepsilon = 0, \hat{\mu} = 1.0 \times 10^{-5}$)
Figure 7: Instability regions with varying dimensionless structural damping $\hat{C}_s$ ($\beta = 100$, $\hat{S}_r = 100$, $\varepsilon = 0$, $\hat{\mu} = 1.0 \times 10^{-5}$)

Figure 8: Instability regions with varying added mass parameter $\beta$ and dimensionless structural damping $\hat{C}_s$ ($\hat{S}_r = 100$, $\varepsilon = 0$, $\hat{\mu} = 1.0 \times 10^{-5}$)
4 Conclusions

In this paper, we examined the stability of the web axially moving in the fluid-filled narrow space with shear fluid flow and the instability regions for various parameters. As a result, the analytical results showed that the flutter type instability and divergence type instability occur in the axially moving web due to the shear fluid flow, and clarified the variation of the instability region with varying the added mass parameter, tensile force and viscous damping.

References