The bridged edge crack for a fibre reinforced solid with localized damage
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Abstract

We consider the problem of an edge crack of finite length which is located normal to the boundary of a uniformly stressed plate. The region in the vicinity of the crack is assumed to be damaged over an elliptical region and the fibres exhibit continuity or bridging across the faces of the crack. The paper outlines the use of a boundary element technique for the determination of the influence of the damaged zone and the crack bridging on the stress intensity factor at the crack tip.

1 Introduction

Unidirectionally reinforced solids are utilized quite extensively in engineering applications where directional strength can be used to advantage. The applications can range from filament winding of pressure vessels to unidirectionally reinforced structural plates used in the aerospace industry. Usually, such composites consist of a relatively brittle matrix which is reinforced with a set of elastic fibres with significantly higher strength and stiffness characteristics. This particular attribute of unidirectionally reinforced composites results in challenging problems which range from the micro-mechanical to the structural level. Among these, fracture initiation is regarded as an important aspect which influences the integrity of the entire composite. Fracture in unidirectionally reinforced composite materials can be initiated by a variety of micro-mechanical processes, including matrix cracking, matrix micro-cracking, matrix yield, fibre pull-out, matrix–fibre delamination and matrix damage, which lead to unacceptable performance in unidirectionally reinforced composites [see, e.g., Backlund1 and Selvadurai and Au2].

The present paper deals with the problem of a unidirectionally reinforced plate in which an edge crack has occurred in the matrix of a region with fibre–matrix debonding. The fibre–matrix debonding is essentially viewed as a damaged zone in which the fibres act separately without the advantage of continuity at the fibre–matrix interface. Such delaminations can occur as a result of dynamic loads, impact loads and localized thermal loading of the composite. Other factors, including surface crazing, can also lead to the through-thickness
cracking and attendant fibre–matrix delaminations in the vicinity of the matrix crack. The process of flaw and crack bridging occurs as a result of fibres which maintain continuity across the matrix crack [see, e.g., Kelly3, Beaumont4, Selvadurai5]. The theoretical modelling of bridged cracks has been considered by many investigators. Selvadurai6,7,8 examined the problem of bridging of penny-shaped cracks and spheroidal flaws located in unidirectionally reinforced solids.

In this paper we consider the bridged edge crack problem for a unidirectionally reinforced plate and incorporate a fibre-matrix debonded region which extends symmetrically about the edge crack. The specific form of the fibre matrix debonded region is assumed to be semi-elliptical. Owing to the complexity of the geometry associated with the bridged edge crack, the problem is not amenable to convenient analytical solution. A boundary element procedure is employed to evaluate the stress intensity factor for a bridged edge crack, and establishes the moderating influence of the elliptical fibre debonded region.

2 Governing equations

The elastic analysis of unidirectionally reinforced solids can be approached by idealizing such solids as transversely isotropic elastic materials. Extensive accounts of the applications of the theory of elasticity for transversely isotropic materials are given by Green and Zerna9, Lekhnitskii10, Spencer11 and Christensen12. The bulk properties of the transversely isotropic elastic idealization can be estimated by recourse to the theory of mixtures applicable to fibre reinforced solids13. We assume that the unidirectionally reinforced plate can be modelled as an orthotropic elastic plate where a principal axis coincides with the fibre direction. The constitutive relationships governing the elastic response can be written as

\[
\sigma_{\alpha\nu} = c_{11}\epsilon_{\alpha\nu} + c_{12}\epsilon_{\nu\nu}; \quad \sigma_{\nu\nu} = c_{12}\epsilon_{\alpha\nu}c_{12}\epsilon_{\nu\nu}; \quad \sigma_{\nu\nu} = 2c_{66}\epsilon_{\nu\nu},
\]

where \(\sigma_{\alpha\nu}, \sigma_{\nu\nu}\) and \(\sigma_{\nu\nu}\) are the in-plane Cauchy stresses, \(\epsilon_{\alpha\nu}, \epsilon_{\nu\nu}\) and \(\epsilon_{\nu\nu}\) are the corresponding strains and \(c_{ij}\) are the effective elastic constants. Using the results given by Christensen12 we can write \(c_{ij}\) as

\[
c_{11} = \bar{K} + \bar{G}; \quad c_{12} = 2\bar{\nu}\bar{K}; \quad c_{22} = \bar{E} + 4\bar{\nu}^2\bar{K}; \quad c_{66} = \bar{G}.
\]

where \(\bar{K}, \bar{G}, \bar{E}, \bar{\nu}\) and \(\bar{G}\) depend on the elastic constants of the fibre (f) and matrix (m), the volume fraction \((V_f)\) of the fibres, etc. Specific expressions for \(\bar{K}, \bar{G}\) etc., are given by Christensen12. In the absence of body forces, the Navier–Cauchy equations of equilibrium for an orthotropic elastic material are given by

\[
c_{11}\frac{\partial^2 u_1}{\partial x^2} + (c_{12} + c_{66})\frac{\partial^2 u_1}{\partial x \partial y} + c_{66}\frac{\partial^2 u_1}{\partial y^2} = 0
\]

\[
c_{22}\frac{\partial^2 u_1}{\partial y^2} + (c_{12} + c_{66})\frac{\partial^2 u_1}{\partial x \partial y} + c_{66}\frac{\partial^2 u_1}{\partial x^2} = 0
\]

where \(u_1\) and \(u_1\) are the in-plane displacements.
3 The boundary element method

The boundary integral equation for an orthotropic elastic continuum can be written as

\[
\vec{c}_j(P)\vec{u}_j(P) + \int_S \left\{ \vec{t}_j(P,Q)\vec{u}_j(Q) - \vec{u}_j(P,Q)\vec{t}_j(Q) \right\} dS = 0
\]  

where \( P \) is the point under consideration, \( Q \) is a general point, \( S \) is the boundary of the composite region, \( \vec{c}_j \) are constants, \( \vec{t}_j \ (j=x,y) \) are components of the traction, \( \vec{u}_j \ (j=x,y) \) are components of the displacements and \( \vec{u}_j^* \) and \( \vec{t}_j^* \) are, respectively, the displacement and traction fundamental solutions. The fundamental solutions take the forms (see, e.g., Rizzo and Shippy\textsuperscript{13})

\[
\vec{u}_{xx}^* = \frac{q_1q_2}{2\pi c_{66}(q_1 - q_2)} \left\{ \frac{\gamma_1}{q_1} \ln \left( x^2 + y_1^2 \right)^{1/2} - \frac{\gamma_2}{q_2} \ln \left( x^2 + y_2^2 \right)^{1/2} \right\}
\]

\[
\vec{u}_{yy}^* = \frac{-1}{2\pi c_{66}(q_1 - q_2)} \left\{ \frac{q_1}{\gamma_1} \ln \left( x^2 + y_1^2 \right)^{1/2} - \frac{q_2}{\gamma_2} \ln \left( x^2 + y_2^2 \right)^{1/2} \right\}
\]

\[
\vec{u}_{xy}^* = \frac{-q_1q_2}{2\pi c_{66}(q_1 - q_2)} \left\{ \tan^{-1} \left( \frac{y_1}{x} \right) - \tan^{-1} \left( \frac{y_2}{x} \right) \right\}
\]

e.tc., and

\[
\vec{t}_{xx}^* = \frac{-q_1q_2}{2\pi(q_1 - q_2)} \left\{ \frac{(1 + q_1)(x + y_1)}{\gamma_1, q_1(x^2 + y_1^2)} - \frac{(1 + q_2)(x + y_2)}{\gamma_2, q_2(x^2 + y_2^2)} \right\}
\]

\[
\vec{t}_{yy}^* = \frac{1}{2\pi(q_1 - q_2)} \left\{ \frac{(1 + q_1)(x + y_1)}{\gamma_1(x^2 + y_1^2)} - \frac{(1 + q_2)(x + y_2)}{\gamma_2(x^2 + y_2^2)} \right\}
\]

\[
\vec{t}_{xy}^* = \frac{-q_1q_2}{2\pi(q_1 - q_2)} \left\{ \frac{(1 + q_1)(y_1 + y_1)}{q_1(x^2 + y_1^2)} - \frac{(1 + q_2)(y_1 + y_2)}{q_2(x^2 + y_2^2)} \right\}
\]

e.tc., where

\[
q_i = \frac{c_{11}\gamma_i^2 - c_{66}}{c_{12} + c_{66}}
\]

and \( \gamma_i \) are the roots of the characteristic equation

\[
c_{11}c_{66}\gamma^4 - [c_{12}(c_{12} + 2c_{66}) - c_{11}c_{22}]\gamma^2 + c_{22}c_{66} = 0
\]

Also, in (6) and (7) the geometric terms are

\[
x = x(P) - x(Q); \quad y = y(P) - y(Q); \quad y_i = y/\gamma_i
\]

and \((n_x, n_y)\) are the direction cosines of the outward unit normal to \( S \). In developing the fundamental solutions (6) and (7) it is implicitly assumed that \( \gamma_i \)}
are both real and positive. This assumption is satisfied by most fibre reinforced composite materials.

In the boundary element modelling, the boundary \( S \) is divided into \( M \) segments, such that for a single element the geometry is given by

\[
[x; y] = \sum_{\beta=1}^{3} N_{(p)}^{(\beta)}(\chi^{(\beta)}; y^{(\beta)})
\]

(11)

and the variables \( u_i \) and \( t_i \) can be written as

\[
[u_i; t_i] = \sum_{\beta=1}^{3} N_{(p)}^{(\beta)}(u_i^{(\beta)}; t_i^{(\beta)}) = [N(\zeta)]\{u_i\} \{t_i\}
\]

(12)

The interpolation functions are

\[
N_{(p)}^{(1)} = \frac{\zeta(\zeta - I)}{2}; \quad N_{(p)}^{(2)} = (I - \zeta^2); \quad N_{(p)}^{(3)} = \frac{\zeta(\zeta + I)}{2}
\]

(13)

with \(-I \leq \zeta \leq I\). For the discretized boundary the boundary integral equation (5) can be written as

\[
\tilde{c}_{ik}u_k + \sum_{e=1}^{M} \int t^*_k [N(\zeta)] |J| d\zeta \{u_k\}^e
\]

\[
= \sum_{e=1}^{M} \int u^*_k [N(\zeta)] |J| d\zeta \{t_k\}^e
\]

(14)

where \( e \) is the element number and \(|J|\) is the Jacobian given by

\[
|J| = \left[ \left( \frac{\partial x}{\partial \zeta} \right)^2 + \left( \frac{\partial y}{\partial \zeta} \right)^2 \right]^{1/2}
\]

(15)

Upon completion of the integrations of the coefficients occurring in (14) the complete system equation for every location \( P \) can be written as

\[
[H] \{u\} = [G] \{t\}
\]

(16)

where \([H]\) and \([G]\) are the corresponding coefficients matrices derived from the fundamental solutions.

4 The bridged zone

We consider the matrix crack which is aligned normal to the edge of a boundary of a unidirectionally reinforced elastic plate. The matrix crack occurs in a zone where fibre-matrix debonding is present. For convenience we shall assume that the zone of debonding has a semi-elliptical shape as shown in Figure 1. The plate is subjected to a uniform far field stress \( \sigma_0 \) in the direction of the fibres. The fibres are continuous in the debonded zone and exert a displacement dependent traction constraint at the boundary of the debonded zone. This, in
Damage and Fracture Mechanics

... influences the stress intensity factor which develops at the tip of the crack. We model the debonded fibres as a series of linear springs of variable length and the effective lengths required to compute the stiffness of the springs are determined from the dimensions of the debonded region. When the debonded region is symmetrically located across the crack (Figure 1) the traction boundary conditions are

\[ t_y + \frac{E_f V_f \Delta u_s}{y_0 (x)} = 0; \quad t_s = 0 \]  

(17)

Incorporating the boundary condition (18) in (16) we have

\[ \begin{bmatrix} H_1 & (H_2 + G_2 \frac{E_f V_f}{y_0}) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} G_1 \end{bmatrix} \{ t_1 \} \]  

(18)

where the subscript 2 refers to the boundary on which (18) is prescribed, and the subscript 1 refers to the remainder of the boundary.

5 Modelling of crack tip

The analytical results for the stress field at the tip of cracks located in isotropic and transversely isotropic materials are given by Rooke and Cartwright\(^1\). In the special case when the crack is oriented along the x-direction, the stress \( \sigma_{yy} \) is given by

\[ \sigma_{yy}(r, \theta) = \frac{K_I}{\sqrt{2\pi r}} f_{yy}(\theta) \]  

(19)

where \((r, \theta)\) are the local coordinates at the crack tip, \( f_{yy}(\theta) \) is an angular distribution function with \( f_{yy}(0) = 1 \) and \( K_I \) is a stress intensity factor. It is evident that \( \sigma_{yy} \) exhibits a \( 1/\sqrt{r} \) type stress singularity at the crack tip. To model this stress singularity in the boundary element scheme, the singular traction-quarter point element proposed by Cruse and Wilson\(^1\) is utilized. The displacements and tractions are given by

\[ u_i = b_0 + b_1 \sqrt{r} + b_2 r; \quad t_i = c_0 \frac{\sqrt{r}}{r} + c_1 + c_2 \sqrt{r} \]  

(20)

where \( b_n \) and \( c_n \) \((n = 0, 1, 2)\) are constants. The accuracy of the boundary element scheme has been verified by a number of investigators\(^1\). A detailed account of recent developments is given by Aliabadi\(^1\). The crack opening mode stress intensity factor can be obtained from the result

\[ K_I = \lim_{x \to 0} \sqrt{2\pi x} \sigma_{yy}(x, 0) \]  

(21)
6 Numerical results

The unidirectional reinforced plate containing the matrix edge crack, with an enclosing fibre debonded region which provides fibre continuity, is shown in Figure 1. The extent of the debonded region is assumed to be semi-elliptical. The boundary element technique is used to evaluate the stress intensity factor at the tip of the matrix crack in the presence of fibre debonding and attendant fibre continuity across the debonded region. In the region exterior to the debonded boundary, the fibre reinforced plate has orthotropic elastic properties consistent with the intact plate; within the debonded region the plate is modelled as an orthotropic elastic material without fibres (i.e., $E_f - 0$) and the continuous fibres which bridge the boundary of the debonded region are modelled as linear elastic elements. The parameters influencing the stress intensity factor for the bridged edge crack includes (i) the fibre–matrix modular ratio ($E_f/E_m$); (ii) the Poisson’s ratios for the fibres and matrix phases ($\nu_f, \nu_m$); (iii) the volume fraction of the fibre and matrix phases ($V_f, V_m$); (iv) the geometry of the elliptical fibre debonded region ($b/c$); (v) the extent of matrix cracking in the fibre debonded region ($\sigma_0$).

In the numerical treatment of the problem we assume that $\nu_f = \nu_m = 0.20$. For ease of presentation we normalize the computed stress intensity factors in relation to the stress intensity factor $K_i^0$ for a line crack (of length $2a$) located in an isotropic plate of infinite extent and subjected to a uniform stress $\sigma_0$, i.e.

$$K_i^0 = \sigma_0 \sqrt{\pi a}$$  \hspace{1cm} (22)

Figures 2 and 3 illustrate the variation of the normalized stress intensity factor at the tip of the matrix crack $K_i/K_i^0$ as a function of the fibre–matrix modular ratio ($E_f/E_m$), the extent of the debonded region ($b/c$) and the geometry of the crack in relation to the geometry of the debonded region. It is evident that the fibre continuity has a significant influence in attenuating the stress intensity factor at the crack tip. As the fibre–matrix modular ratio increases the stress intensity factor is suppressed. The numerical results indicate that when $(E_f/E_m) > 10^5$, the stress intensity factor is virtually zero. The geometry of the damaged delaminated region also has an influence on the stress intensity factor; as the aspect ratio of the debonded region $b/c$ increases, there is a corresponding increase in the stress intensity factor. As the fibre elasticity decreases (i.e., as $(E_f/E_m) \rightarrow 0$, the stress intensity factor reduces to that applicable to an edge crack located in an orthotropic elastic material without fibres, i.e., $(K_i/K_i^0) \rightarrow 1.12$ (see, e.g., Rooke and Cartwright14).

7 Conclusions

The paper presents an elementary model of the behaviour of bridged edge crack located in a unidirectionally reinforced plate. The model also accounts for the influence of a damaged region where fibre debonding has occurred over a semi-elliptical region. The boundary element technique can be used quite effectively to examine the influence of fibre continuity and localized damage on the stress intensity factor for a matrix crack located symmetrically within the debonded
region. The numerical results confirm the trends observed in purely mathematical treatments of penny-shaped bridged cracks located in fibre reinforced solids of infinite extent. A limitation of the modelling is the a priori specification of the fibre debonded region. The computational methodologies can be extended to include an incremental analysis where fibre debonding will evolve in a non-uniform or irregular fashion when matrix cracking transfers the load directly to the fibres. The results of the computational modelling carried out in connection with this paper, however, illustrates the significant importance of a localized matrix damage region, in the form of fibre-matrix debonding, and fibre bridging across a damaged zone in attenuating the crack opening mode stress intensity factor for the edge crack.

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References


Figure 1. Bridged edge crack in a unidirectional fibre reinforced plate.
Figure 2. Variation in the normalized stress intensity factor at the tip of the bridged crack.

Figure 3. Variation in the normalized stress intensity factor at the tip of the bridged crack.