Mathematical model for modelling the cyclic hardening/softening transient response of metallic materials

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Abstract

A mathematical model for modelling the cyclic hardening/softening transient response of metallic materials is presented in this paper. The variation of stress amplitude in the stress-strain response (by strain control) is quantitatively analysed. A corresponding equation for modelling the variation of the stress amplitude is given. The model demonstrates that the variation of the stress amplitude deals with not only the material properties and the reversal number of material deformation into the plastic range, but also the strain amplitude. Comparing with experimental results, the model presented in this paper is better than the other models. Moreover the model has no need of any experimental parameters such as the cyclic hardening/softening coefficient fitted by a lot of experimental data, so plenty of payout can be saved. The variation of the cyclic strength coefficient and the cyclic hardening exponent during transient response are also analysed by using the model. It is found that the variations of both the cyclic strength coefficient and the cyclic hardening exponent are great in the first half of the fatigue life, but little in the second half, which is in accordance with the common experimental results of the stress-strain response of metallic materials.

1 Introduction

In the past decades, the local stress-strain approach has been widely applied in the low/medium cycle fatigue life prediction of engineering metallic structures
[1-3]. Usually, the stable cyclic stress-strain response (conventionally called the cyclic stress-strain curve) is employed in the fatigue life prediction by the local stress-strain approach. This is because no effective and simple method has been presented in previous work to take into account the cyclic hardening/softening transient response of the stress-strain relation. Actually, the effect on the result of fatigue life prediction has not been correctly found by using the stable cyclic stress-strain curve instead of the transient cyclic stress-strain curve. As we know, the phenomenon of cyclic hardening/softening really exists in a lot of materials, especially in metallic materials. The objective of this paper is to develop a mathematical model for modelling the cyclic hardening/softening transient response of metallic materials, and to provide a basis for the exploration of the effect of the transient cyclic stress-strain response on fatigue life prediction.

2 Cyclic hardening/softening characteristics

Under cyclic loading, the resistance of deformation of metallic materials continuously changes with the loading cycles. The transient phenomena are classified into two types and are designated as cyclic hardening and cyclic softening under the condition of strain control. As illustrated in Figure 1, cyclic hardening or cyclic softening is denoted by a cycle-dependent increase or decrease in stress range under constant strain range cycling.

![Cyclic hardening and softening](image)

(a) Cyclic hardening  
(b) Cyclic softening

Figure 1: The characteristics of cyclic hardening/softening.
Moreover, under constant stress range cycling, the transient phenomena would exhibit the other two characteristics: cyclic creep and cyclic relaxation. It can be considered that the phenomena of cyclic creep/relaxation are only different exhibitions of the relative transient characteristics of the cyclic hardening/softening of metallic materials. This kind of viewpoint will be further discussed in another paper. The transient phenomena of cyclic hardening and cyclic softening are concentrated in following investigation.

Usually, the phenomena of cyclic hardening and cyclic softening will decrease with the increase of cycles. At about 20–50% of the total cycles to fatigue failure, a state of saturation is approached and a hysteresis loop is formed. A hysteresis loop has two branches: the upper branch corresponds to the tensile loading direction, and the lower branch corresponds to the compressive direction. During the transient conditions, a hysteresis loop is not closed, and the hysteresis branches are not identical. However, in the saturated state, the hysteresis loop is fully closed, and the two hysteresis branches are identical, one being the mirror image of the other. Each branch is designated as a “reversal”, and the cyclic progress is denoted in terms of the number of reversals.

Obviously a series of different sized hysteresis loops can be obtained under the control of different strain ranges. A stable cyclic stress-strain curve can be obtained by connecting all of the tips of the hysteresis loops. In most of the past engineering cases, only these kinds of stable cyclic stress-strain curves are applied in the fatigue life prediction of metallic structures, while the transient characteristics of materials are ignored. Normally, the cyclic stress-strain of a metallic material is obtained by the test of small smooth specimens under the control of different strain ranges. Most of the cyclic stress-strain curves of structural metals are symmetric about the original point of \(\sigma-\varepsilon\) coordinates. We shall focus the following discussion only on this kind of material.

3 Mathematical model of transient response

An outstanding work on the quantitative analysis of the transient cyclic softening/hardening characteristics of metallic materials is that by Jhansale [4]. It was considered that the transient changes in the hysteresis branches are essentially caused by the changes in length of the initial “linear” elastic parts. The slope of the “linear” elastic portions and the shape of the nonlinear portions remain virtually unchanged. The change in the elastic part is designated as the “yield range increment” (YRI), since it denotes a change in the yield range of the material. The yield range is twice the yield strength because of the scale factor of two between the monotonic and hysteretic stress-strain paths. Thus, transient phenomena can be described in terms of a simple parameter YRI or “yield strength increment” (YSI), which is one-half of YRI. The modulus can be assumed to remain unchanged during cyclic loading. The invariant nonlinear portion of the hysteresis branch that is unaffected by the cyclic plastic straining represents an intrinsic material property. Denoting such an intrinsic stress-strain curve as the “skeleton stress-strain curve”, the invariant nonlinear hysteresis
branch will be geometrically similar to the skeleton stress-strain curve, but magnified by a scale factor of two. To measure the variation of YRI during cyclic hardening/softening, the transient properties of several metals such as 2024-T4 aluminium, SAE-4340 steel and A-36 steel were investigated by testing. The following general expression for the rate of change of yield strength was assumed for analysing the test data:

\[
\frac{d\sigma_y}{dR} = C_{HS} \left( \frac{\sigma_{ys} - \sigma_y}{R^p} \right)^m
\]

(1)

where \( d\sigma_y / dR \) is the rate of change of yield strength per reversal; \( \sigma_y \) is the current yield strength; \( \sigma_{ys} \) is the saturation value of yield strength; \( R \) is the current reversal number; \( C_{HS} \) is the coefficient of hardening or softening; and \( m \) and \( p \) are suitable exponents. For the material 2024-T4 aluminium, \( C_{HS} \), \( m \) and \( p \) can respectively be taken as 0.1, 1 and 1. An equation relating yield strengths and the number of reversals can be obtained by integrating Equation (1):

\[
\int_{\sigma_{ys}}^{\sigma_y} \frac{d\sigma_y}{\sigma_{ys} - \sigma_y} = C_{HS} \int_{1}^{R} \frac{dR}{R}
\]

(2)

where \( \sigma_{ys} = \sigma_s \), the material yield strength corresponding to monotonic tensile (i.e. 1 reversal). Thus we obtain the following equation from Equation (2):

\[
\Delta\sigma_{yR} = \Delta\sigma_{ys} \left( 1 - R^{-C_{HS}} \right)
\]

(3)

where \( \Delta\sigma_{yR} \) is the current increment of yield strength under \( R \) reversals, and \( \Delta\sigma_{ys} \) is the constant increment of yield strength under saturation conditions. It should be mentioned that both \( \Delta\sigma_{yR} \) and \( \Delta\sigma_{ys} \) denote only the magnitude, not the sign of the increments. It is found that when \( R = 1 \), \( \Delta\sigma_{yR} = 0 \); when \( R \to \infty \), \( \Delta\sigma_{yR} \to \Delta\sigma_{ys} \), where it follows the concept of “saturated stress-strain curve”.

A view of cyclic hardening “without saturation” was presented for LY12-CZ aluminium (a material equivalent to 2024-T4 aluminium) in [5], in which the concerned test data of the material are analysed and the expression of the yield strength increment is provided as:

\[
\Delta\sigma_{yR} = C_{HS} \log R
\]

(4)

where \( C_{HS} = 46.5 \) MPa for LY12-CZ aluminium, and \( C_{HS} = 56.8 \) MPa for 2024-T4 aluminium. Two curves of the \( \Delta\sigma_{yR} - \log R \) relation for LY12-CZ aluminium and 2024-T4 aluminium are obtained by Equation (3) and Equation (4), respectively, and shown in Figures 2 and 3. The responding test data are also
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![Graph showing the relationship between \( \Delta \sigma_{yR} \) and \( \lg R \) for LY12-CZ.](image)

Figure 2: Test results and calculated curves of \( \Delta \sigma_{yR} - \lg R \) for LY12-CZ.

...demonstrated in Figures 2 and 3 for comparison. It is easy to find that it is impossible to fit all of the data by one curve, no matter what, by Equation (3) or by Equation (4). In fact it was pointed out in [4] that the coefficient of hardening or softening \( C_{HS} \) deals with not only material properties, but also the magnitude of strain amplitude (or strain range).

Reviewing the test data in Figures 2 and 3, we can find that the data under the identical strain control distribute approximately along one straight line, although all of data under diverse strain control scatter out. For more cases, we can find similar results in [6], in which the curves of stress amplitude – logarithm of reversal number (i.e. \( \sigma_a - \lg R \) curves) under the loading control of different strain ranges are presented for four metallic materials: 2024-T4 aluminium, 7075-T6 aluminium, SAE-4340 steel and Ti-8Al-1Mo-1V titanium. Almost all the curves demonstrate one straight line under the control of constant strain. Based on the above analysis, the relation of \( \sigma_a - \lg R \) under the control of a strain range is assumed as a straight line in this paper, and described as follows:

\[
\sigma_a(R) = A + B \lg R
\]
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![Graph showing test results and calculated curves of $\Delta \sigma_{yR} - \lg R$ for 2024-T4.](image)

Figure 3: Test results and calculated curves of $\Delta \sigma_{yR} - \lg R$ for 2024-T4.

where $A$ and $B$ are factors to be determined. The following conditions are introduced:

$$
\begin{align*}
\text{when} & \quad R = 1, \quad \sigma_a = \sigma_{a1} \\
\text{when} & \quad R = N_f, \quad \sigma_a = \sigma_{aN}
\end{align*}
$$

(6)

where $\sigma_{a1}$ is the initial stress amplitude (the amplitude value corresponding to the monotonic stress-strain curve) under the control of a strain range; $\sigma_{aN}$ is the stress amplitude at $N_f$ reversals (the amplitude value corresponding to the cyclic stress-strain curve) under the control of a strain range. $N_f$ is the number of cycles to fatigue failure. Since one cycle is equal to two reversals, the number of reversals to fatigue failure is $2N_f$. Substituting Equation (6) into Equation (5), we have:

$$
\sigma_a(R) = (\sigma_{aN} - \sigma_{a1}) \lg R / \lg N_f
$$

(7)
where \( R \) denotes the reversal number of material deformation into the plastic range; \( \sigma_{a1}, \sigma_{aN} \) and \( N_f \) reflect the material properties and the magnitude of strain ranges. Equation (7) is the basic expression of the mathematic model of modelling the cyclic hardening/softening transient response of the metallic materials presented in this paper.

From Equation (7), the cyclic hardening is “without saturation”, and the variation of stress amplitudes within \( N_f < R \leq 2N_f \) is much smaller than that within \( R \leq N_f \). Let \( \Delta \sigma_a(R) = \sigma_a(R) - \sigma_{a1} \) then

\[
\Delta \sigma_a(R) = (\sigma_{aN} - \sigma_{a1}) \log R / \log N_f
\]

Comparing Equation (8) with Equation (4), \( \Delta \sigma_a(R) \) is identical to \( \Delta \sigma_{yR} \) according to the definitions of themselves; the term \( (\sigma_{aN} - \sigma_{a1}) / \log N_f \) in Equation (8) corresponds to \( C_{HS} \) in Equation (4). The term of \( (\sigma_{aN} - \sigma_{a1}) / \log N_f \) is a variable, whereas \( C_{HS} \) is only a constant value.

The curves obtained by Equation (8) are also illustrated in Figures 2 and 3 (see the solid curves) in order to compare with the results from Equation (3) and Equation (4). It is easy to find that the result from Equation (8) is more reasonable than those from Equations (3) and (4). In the calculation, \( \sigma_{a1} \) and \( \sigma_{aN} \) are obtained by the iteration of the following two Equations respectively:

\[
\varepsilon_a = \frac{\sigma_{a1}}{E} + \left( \frac{\sigma_{a1}}{K} \right)^{\frac{1}{n}}
\]

\[
\varepsilon_a = \frac{\sigma_{aN}}{E} + \left( \frac{\sigma_{aN}}{K'} \right)^{\frac{1}{n'}}
\]

where \( \varepsilon_a \) is the strain amplitude; \( E \) is the elastic modulus; \( K \) is the strength factor; \( n \) is the strain hardening exponent; \( K' \) is the cyclic strain hardening factor; and \( n' \) is the cyclic strain hardening exponent.

It is worth pointing out that no coefficient like \( C_{HS} \) is needed in Equation (8). Therefore, lots of tests for obtaining this kind of coefficient can be exempted and plenty of payout can be saved.

### 4 Analysis for transient variation of \( K(R) \) and \( n(R) \)

From Equations (9) and (10) we can deduce that the transient cyclic strength factor and the transient strain hardening exponent should be continuously
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changed during transient hardening/softening response. The variation of \( K(R) \) is also observed in the above analysis of \( \sigma_a(R) \); here a similar expression is assumed for \( K(R) \):

\[
K(R) = (K' - K) \log R / \log N_f
\]  
(11)

in this case the \( n(R) \) can be obtained from the following equation:

\[
e_v = \frac{\sigma_a(R)}{E} + \left( \frac{\sigma_a(R)}{K(R)} \right)^{n(R)}
\]  
(12)

That is

\[
n(R) = \frac{\log \left[ \frac{\sigma_a(R)}{K(R)} \right]}{\log \left[ e_v - \sigma_a(R) \right] / E}
\]  
(13)

It is found from Equations (12) and (13) that the changes of \( K(R) \) and \( n(R) \) are great in the first half of the fatigue life (\( R \leq N_f \)), but small in the second half (\( N_f < R \leq 2N_f \)), when \( R = N_f \), \( K(R) \) and \( n(R) \) achieve the values of \( K' \) and \( n' \). Obviously, it is in accordance with the common experimental results of the stress-strain response of metallic materials. For instance, the variations of \( K(R) \) and \( n(R) \) are listed in Table 1 for the cyclic hardening material 2024-T4 aluminum and the cyclic softening material SAE-4340 steel. Each of the materials is experimentally controlled under constant strain range cycling.

5 Conclusions

A mathematical model for modelling the cyclic hardening/softening transient response of metallic materials is presented in this paper. The model demonstrates that the variation of the stress amplitude deals with not only the material properties and the reversal number of the material deformation into the plastic range, but also the strain amplitude. Comparing with experimental results, the model presented in this paper is better than others. Moreover, the model has no need of any experimental parameters such as the cyclic hardening/softening coefficient; therefore lots of tests for obtaining this kind of coefficient can be exempted and plenty of payout can be saved. The variation of the cyclic strength coefficient and cyclic hardening exponent during transient response are also analysed by using the model. It is found that the variations of both cyclic strength
coefficient and cyclic hardening exponent are great in the first half of the fatigue life, but small in the second half, which is in accordance with the common experimental results of the stress-strain response of metallic materials.

Table 1: The variations of $K(R)$ and $n(R)$ with $R$.

<table>
<thead>
<tr>
<th>Materials</th>
<th>$R$</th>
<th>$K(R)$ (MPa)</th>
<th>$n(R)$</th>
<th>Parameters</th>
</tr>
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<tbody>
<tr>
<td>2024-T4</td>
<td>1</td>
<td>807.0</td>
<td>0.200</td>
<td>$\varepsilon_a = 0.0182$</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>812.4</td>
<td>0.184</td>
<td>$N_f = 142$ cycles</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>817.8</td>
<td>0.170</td>
<td>$K = 807$MPa</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>824.9</td>
<td>0.153</td>
<td>$K' = 845.6$MPa</td>
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<tr>
<td></td>
<td>20</td>
<td>830.3</td>
<td>0.140</td>
<td>$n = 0.20$</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>835.7</td>
<td>0.129</td>
<td>$n' = 0.11$</td>
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<tr>
<td></td>
<td>100</td>
<td>842.9</td>
<td>0.115</td>
<td></td>
</tr>
<tr>
<td></td>
<td>200</td>
<td>848.2</td>
<td>0.105</td>
<td></td>
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<tr>
<td></td>
<td>284</td>
<td>851.0</td>
<td>0.101</td>
<td></td>
</tr>
<tr>
<td>SAE-4340</td>
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<td>1579.0</td>
<td>0.0660</td>
<td>$\varepsilon_a = 0.029$</td>
</tr>
<tr>
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<td>2</td>
<td>1616.8</td>
<td>0.0762</td>
<td>$N_f = 167$ cycles</td>
</tr>
<tr>
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<td>0.0863</td>
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<tr>
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<td>0.0996</td>
<td>$K' = 1857.8$ MPa</td>
</tr>
<tr>
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<td>0.1096</td>
<td>$n = 0.066$</td>
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<tr>
<td></td>
<td>40</td>
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<td>0.1196</td>
<td>$n' = 0.14$</td>
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Acknowledgement

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References


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