Strain field and strain history of nonproportional loading

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Abstract

Nonproportional multiaxial loading of components is of great interest in fatigue and damage studies. To present a complete strain field and strain history of such loading, an incremental theory of plasticity should be employed. However, due to its incremental nature, the computational time is not favourable. On the other hand the total deformation plasticity is less time consuming but is path independent and hence is not capable of predicting strain history. Also the strain predictions are not correct. In this paper an appropriate path dependent semi total deformation method is presented for nonproportional loading. In the formulation presented both linear and nonlinear stressing is considered. Different examples are considered and results obtained from present method are compared to incremental plasticity using finite element method.

1 Introduction

In classical plasticity, two types of plasticity theories have been used; i.e., total deformation and incremental theory. Hencky [1] proposed a one to one correspondence between current stress and total plastic strain. Analysis on this basis is independent of the path of loading and always assumes a proportional loading between the starting and ending point of loading. On the other hand Prandtl [2] and Reuss [3] proposed a relationship between the increment of plastic strain and current stress. Analysis on this basis is path dependent. For proportional loading, incremental theory leads to Hencky’s formulation [4]. There has been lots of research work regarding the limitations of applicability of total Deformation. Amongst them, Budiansky [5] showed that physical soundness of total deformation is not limited to proportional loading. However,
the difference between incremental and total plasticity increases as the loading tends to be completely nonproportional. Experimental result of Mroz and Olszak [6] showed that plastic strain depend not only on the current stress but also on the history of stressing. Sidebottom [7] showed that in nonproportional loading, incremental theory is in better agreement with experimental result. Therefore incremental plasticity results are more favourable. But this theory due to its incremental nature and long computational time is not favourable. To reduce computational time several approximate methods have been used. These methods have been reviewed by Seeger and Heuler [8]. The most frequently used relation was derived by Neuber [9] and its general form were given by Seeger et al [10], Glinka [11] and Jiang and Xu [12]. Jahed et al [13] presented a new total deformation formulation for analyzing a sequence of linear nonproportional load steps. They present a time efficient and path dependent method for calculating strain distribution at the end of the loads. This formulation in a general form has the following preferences: 1) fast solution; 2) dependency to load path; 3) capability of predicting path dependent strain values; and 4) capability of calculating strain history even in one increment solution. In this paper the method of Jahed et al [13] is extended to more general loading paths. Formulations for piece wise linear and nonlinear loading paths are presented. Different examples are worked out and their results are compared to incremental plasticity results. These examples include thin tube under tension and torsion, shaft with a circumferential notch subjected to torsion and tension, and thick walled cylinder under pressure and torsion.

2 Piece wise linear formulation

A sequence of linear loading OA-AB-BC is shown in Fig 1(a). The stress during the second, i.e., AB or any subsequent linear loading may be defined with respect to the start point of that loading. For loading AB stress can be written as follow:

\[ S_y = S_{y}^A + t \Delta S_y \]  
\[ \Delta S_y = S_y^B - S_y^A \quad \text{and} \quad 0 < t < 1 \]  

where \( S_y^A \) and \( S_y^B \) are the deviatoric components of the stresses at the end points of loading AB and \( t \) is the time parameter. Further the Mises equivalent stress

\[ \sigma_e = \sqrt{\frac{3}{2} S_y S_y} \]  

During the second loading with the aid of Eq (1) may be written in the following form

\[ \sigma_e^2 = t^2 \Delta \sigma_e^2 + t(\Delta \sigma_e^2 + \sigma_e^2 + \sigma_e^2) + \sigma_e^2 \]  

where the change in equivalent stress, \( \Delta \sigma_e \), is defined as

\[ \Delta \sigma_e^2 = \sqrt{\frac{3}{2} \Delta S_y \Delta S_y} \]  

Differentiating Eq (3) with respect to \( t \)
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Figure 1: (a) Piece wise nonproportional linear loading. (b) Nonlinear loading.

\[
d\sigma_e = \frac{1}{2} \left( \frac{2t-1}{t^2} \Delta \sigma_e^2 - \Delta \sigma_e^2 + \Delta \sigma_e^2 \right) dt \tag{5}
\]

and substituting in the Prandtl-Reuss equation

\[
d\varepsilon = \frac{3}{2 E_p} \frac{d\sigma_e}{\sigma_e} S_{ij}
\]

The change in plastic strain during the second loading can now be obtained by integrating the Prandtl-Reuss equation with respect to \( t \) \( 0 < t < \alpha \). Where \( \alpha \) is a point on loading path A to B (Fig-1-a) Therefore,

\[
\Delta \varepsilon = \int_0^\alpha \left[ \frac{3}{4 E_p} \frac{1}{t^2} \left( \frac{2t-1}{t^2} \Delta \sigma_e^2 - \Delta \sigma_e^2 + \Delta \sigma_e^2 \right) \Delta S_{ij} + t \Delta S_{ij} \right] dt \tag{7}
\]

For linear hardening material, where plastic modulus is constant:

\[
\Delta \varepsilon = \lambda \left( \frac{\alpha}{2} \left[ \frac{1}{2} \left( \frac{1+A}{B} \right) \ln C - \frac{B}{2} \left( \tan^{-1}\left( \frac{1+A}{B} \right) + \tan^{-1}\left( \frac{1-A}{B} \right) \right) \right] \Delta S_{ij} + S_{ij} (\ln C) \right) \tag{8}
\]

where

\[
A = \frac{\alpha \sigma_e^2 - \Delta \sigma_e^2}{\Delta \sigma_e^2}; \quad B = \sqrt{\alpha^2 \Delta \sigma_e^2 - (1-A)^2}; \quad C = \frac{\alpha \sigma_e^2}{\Delta \sigma_e^2}; \tag{9}
\]

and

\[
\alpha \sigma_e = \frac{1}{2} \left[ \frac{2}{\sqrt{2}} S_{ij} \Delta S_{ij} \right]; \quad \alpha S_{ij} = S_{ij} + \alpha \Delta S_{ij}; \quad \lambda = \frac{3}{2 E_p} \tag{10}
\]

This formulation yields three cases:

1) \( 0 < \alpha < 1 \) represent strain history during loading.
2) \( \alpha = 1 \) represent strain distribution at the end of the load steps.
3) \( \alpha > 1 \) predict strain values if continuation of loading considered.
3 Second order approximation of stress history

In the previous section stresses varied linearly during loading. Now suppose that the stress components variation with respect to time are nonlinear (as shown in fig 1(b)).

Writing deviatoric stresses in the form of series gives:

\[ S_y = \frac{A}{\tau} \frac{\partial S_y}{\partial t} + \frac{A}{\tau^2} \frac{\partial^2 S_y}{\partial t^2} t^2 + \cdots \cdots \cdots \cdots \cdots \cdots \cdots (11) \]

where \( 0 \leq t \leq 1 \) and \( \tau_A = 0 \)

By neglecting higher order terms, For two points of loadings \( ^CS_y_{\tau=1} \) and \( ^BS_y_{\tau=0} \), Eq(11) takes the following form

\[ ^CS_y = A \frac{\partial S_y}{\partial t} + \frac{B}{\tau} \frac{\partial^2 S_y}{\partial t^2} t^2 \]

\[ ^BS_y = A \frac{\partial S_y}{\partial t} + \frac{B}{\tau} \frac{\partial^2 S_y}{\partial t^2} t^2 \]

Solving these two equations with respect to \( \frac{\partial S_y}{\partial t} \) and \( \frac{\partial^2 S_y}{\partial t^2} \) yield

\[ \frac{\partial S_y}{\partial t} = \frac{A}{\tau} S_y - ^BS_y \frac{\partial^2 S_y}{\partial t^2} t_B \]

\[ \frac{\partial^2 S_y}{\partial t^2} = \frac{2(^BS_y - ^CS_y)^2 + (A - ^CS_y) t_B}{t_B (t_B - 1)} \]

Therefore stress component at any point of the loading may be expressed as

\[ S_y = A \frac{\partial S_y}{\partial t} + \left( \frac{A}{\tau} S_y - ^BS_y \frac{\partial^2 S_y}{\partial t^2} t_B \right) t + \frac{2(^BS_y - ^CS_y) + (A - ^CS_y) t_B}{t_B (t_B - 1)} \]

Assuming \( t_B = 1/2 \), stress state during loading can be written as follow

\[ S_y = A \frac{\partial S_y}{\partial t} + \left( ^CS_y + 4 ^BS_y - ^CS_y \right) t + 2t^2 \frac{A - ^CS_y}{2 (t_B - 1)} \]

Differentiation of equivalent stress (Eq 2) yields

\[ d\sigma = \frac{\partial \sigma}{\partial S_y} \frac{\partial S_y}{\partial t} dt = \frac{3}{2} \frac{(S_{en})}{(S_y S_{en})} \frac{S_y S_{en}}{S_{en}} \]

To obtain changes in plastic strain between the end points of the loading, numerical integration of the Prandl-Reuss relation is to be used

\[ \Delta \varepsilon_y = \int_0^1 \frac{3}{2} \frac{1}{E_p} \frac{d\sigma}{\sigma_y} S_y = \int_0^1 \frac{3}{2} \frac{1}{E_p} \frac{3}{2} \frac{(S_{en})}{(S_y S_{en})} S_y dt \]

\[ (19) \]
Figure 2: (a) Shaft under tension and torsion. (b) Three load paths applied on model.

4 Applications

To show the application of formulations presented, different examples are worked out.

4.1 Examples for strain field calculation at the end of the loading

For a linear hardening shaft with a circumferential notch under combined tension-torsion, the stress and strain fields at the end of the second loading (point (B) in figure 2(b)) are not known a priori. Using Prandtl-Reuss relation stresses and strains may be calculated using FEM. Two different loading steps are employed:

1) One increment for the first loading (i.e., path OA or OC) and one increment for second loading (i.e. path AB or CB).
2) Many increments for both load steps. This gives incremental result.

A third solution may be obtained by using the result of 1st solution and the proposed formulation (9). Example presented here show that the strain field which is calculated by this method is much closed to the strained computed by the incremental method. In Tables (1) and (2) strains calculated by proposed method for a thin-tube under tension and torsion are compared to incremental methods.

Also in figures (3), (4) and (5) plastic strains are plotted against the radius at the notched section of the shaft.
4.2 Examples for strain history (Eq (9) for $0 < \alpha < 1$)

To have a complete strain field and history of a nonproportional loading, one need to use large number of increments during loading. However, using this formulation, with just one increment, strain history can be achieved.

![Figure 3: Plastic shear strain comparison.](image)

As an example a thin-tube under tension-torsion is presented. In this example, the thin-tube reaches to the yield point at the end of the first load step and in figures (6) and (7) plastic strain history of the relevant thin-tube are plotted against time($0 < \alpha < 1$), during second loading. Also as another example a thick-walled cylinder under pressure and torsion is presented. It can be seen that by increasing number of increments, and using incremental theory, strain history curves lead to curves obtained by proposed method, which uses just one increment.

![Figure 4: Plastic axial strain comparison.](image)
Figure 5: Plastic radial strain comparison.

Figure 6: History of the plastic strain during second loading (torsion) of a thin tube under tension + torsion; (a) history of the plastic shear strain; (b) history of the Plastic axial strain.

Table 1: Error comparison of the plastic strain calculated at the end of the loading for thin-walled cylinder under tension-torsion by incremental method and proposed method with respect to exact solution [13].

<table>
<thead>
<tr>
<th>Error %</th>
<th>$\varepsilon_{\text{axial}}$</th>
<th>1 Increment</th>
<th>3 Increments</th>
<th>5 Increments</th>
<th>10 Increments</th>
<th>1 Increment (Proposed method)</th>
</tr>
</thead>
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<tr>
<td>$\gamma$</td>
<td>34.5</td>
<td>16</td>
<td>10.4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Error comparison of the plastic strain calculated at the end of the loading for thin-walled cylinder under torsion tension by incremental method and proposed method with respect to exact solution [13].

<table>
<thead>
<tr>
<th>Error %</th>
<th>$\varepsilon_{\text{axial}}$</th>
<th>1 Increment</th>
<th>3 Increments</th>
<th>5 Increments</th>
<th>10 Increments</th>
<th>1 Increment (Proposed method)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>15.5</td>
<td>6.4</td>
<td>4</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Figure 7: History of the plastic strain during second loading (tension) of a thin tube under torsion + tension; (a) history of the plastic shear strain; (b) history of the Plastic axial strain.

Figure 8: History of the Plastic strain during second loading (torsion) of a thick-walled cylinder under tension + torsion at radius \( r_0 = 0.4 \), (a) history of the plastic radial strain; (b) history of the plastic shear strain.
4.3 Second order approximation of stress history

In previous examples stress components varied linearly with time. Suppose that the combined Loading (internal pressure and torsion), Applied to a thick walled cylinder varies nonlinearly with respect to time (figure (9)). This loading with the following second order function may be approximated.

For example: \( P(t) = a_1 t^2 + b_1 t + c_1 \) and \( T(t) = a_2 t^2 + b_2 t + c_2 \)

where \( P(t) \) = Internal Pressure
\( T(t) \) = Torsion Torque

\( t \) = Time and \( a_1, b_1, c_1, a_2, b_2, c_2 \) are related load coefficients.

Now second order formulation is recommended to calculate strain fields. In figures (10-12) plastic strains calculated by incremental and proposed method are presented. From these figures, it is clear that by increasing number of increments results would be more close to proposed method.

Figure 9: Second order loading applied to a thick walled cylinder.

Figure 10: Plastic radial strain comparison.
5 Conclusion

Two methods which are capable of handling nonproportional loading are proposed.
1) A formulation for piece wise linear nonproportional loading:
In this case by one increment (as oppose too many increments) in a load step
plastic strain components may be calculated.

To estimate plastic strain field at the end of load step equation (9) with $\alpha = 1$
may be used. Two different examples were presented (a shaft with a circumferential notch and a thin walled cylinder under tension and torsion) The
strain fields resulting from this method are in good agreement with incremental
plasticity results [14].

If plastic strain history of such loading is desired, equation (9) $0 < \alpha < 1$, can
be used a thick walled cylinder under internal pressure-torsion and a thin walled
cylinder under tension and torsion were presented. Strain history resulting from
this method is in good agreement with incremental plasticity results.
2) A formulation for second order nonlinear stress history
In this case by two increments strain field at the end of load step with an appropriate accuracy may be calculated. A thick walled cylinder under such a loading was presented, and results were in good agreement with incremental results.

The proposed methods for estimating the plastic strain field reduces computation time significantly.

References
