The finite-element modelling of electromagnetic fields

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Abstract

The finite-element modeling of electromagnetic fields directly in terms of the electric or magnetic field strength is discussed. Special emphasis is given to avoiding spurious solutions and to obtaining a maximum accuracy in relation to the computational effort required. In this context the implementation of the compatibility relations and the use of edge elements is discussed in detail. We also compare our approach with the more common methods using vector potentials. A numerical example is given showing the versatility and the accuracy of a code based on our views.

1 Introduction

As compared with finite-element methods for most other applications the finite-element modeling of electromagnetic fields is known to be more complicated in at least two different ways.

First, the finite-element modeling of electromagnetic fields requires a technique for taking into account the discontinuities in components of the field strength (or the flux density) across interfaces where the constitutive coefficients jump. Several methods to solve, or circumvent, this difficulty are known in the literature. We propose a method that offers an optimum in terms of computational efficiency. It basically consists of using edge (or face) elements along the interfaces for the expansion of field strengths (or flux densities) and of using nodal elements elsewhere, i.e. in the subdomains where the constitutive coefficients are continuously differentiable with respect to the spatial coordinates.

Secondly, finite-element methods for modeling electromagnetic fields are
haunted by so-called spurious solutions. Spurious solutions are erroneous solutions that may be obtained even when all basic equations are satisfied accurately. The reason for this is that although the basic equations are satisfied accurately, some of their properties, the compatibility relations, may have been lost due to the discretization process. The only way to solve this difficulty is to add the compatibility relations explicitly to the finite-element formulation.

In the present paper we will describe the general philosophy underlying our approach and discuss a number of specific aspects of it for different applications such as time-domain fields, time-harmonic fields and static or stationary fields. We also present a general discussion of the efficiency of our approach as compared with the efficiency of comparable finite-element methods using vector potentials.

2 Basic equations

We shall discuss our approach for the case of time-domain electromagnetic fields in three spatial dimensions. The reader will have no difficulty to obtain the equations for time-harmonic or static fields and the equations for the two-dimensional case from our equations. As the point of departure for our analysis we use the time-domain electromagnetic field equations

\[ \partial_t (\epsilon \cdot E) + \sigma \cdot E - \nabla \times H = - J^{imp}, \]

\[ \partial_t (\mu \cdot H) + \nabla \times E = - K^{imp}, \]

where \( J^{imp} \) and \( K^{imp} \) are imposed sources of electric and magnetic current that are known, throughout the domain of computation \( D \) (see Fig. 1), as a function of the time coordinate \( t \). The field equations are supplemented with the interface conditions

\[ \nu \times E \text{ and } \nu \times H \text{ continuous across the interface } \mathcal{I}, \]

where \( \nu \) is the unit vector along the normal to the interface \( \mathcal{I} \), and the boundary conditions

\[ \nu \times E = \nu \times E^{ext} \text{ on } \partial D_E \text{ and } \nu \times H = \nu \times H^{ext} \text{ on } \partial D_H, \]

where \( \nu \) is the unit vector along the normal to the outer boundary \( \partial D = \partial D_E \cup \partial D_H \) (with \( \partial D_E \cap \partial D_H = \emptyset \)) of the domain of computation \( D \), and where \( \nu \times E^{ext} \) and \( \nu \times H^{ext} \) are known, along the relevant parts of this outer boundary, as a function of \( t \). Together with the initial conditions at \( t = t_0 \), (1)-(4) define an electromagnetic-field problem with a unique solution, see Stratton [1].

A time-domain finite-element method may be based on (1) and (2), or on the second-order equation for the electric field strength

\[ \partial_t^2 (\epsilon \cdot E) + \partial_t (\sigma \cdot E) + \nabla \times (\mu^{-1} \cdot \nabla \times E) = - \partial_t J^{imp} - \nabla \times (\mu^{-1} \cdot K^{imp}), \]
that is obtained by eliminating the magnetic field strength from these equations. For deriving (5) we assumed that the permeability tensor does not depend on time. A frequency-domain finite-element method may be based on the frequency-domain versions of (1) and (2), or on one of the second-order equations that is obtained by eliminating either the magnetic or the electric field strength from them. A finite-element method for static and stationary fields can only be based on the static limits of (1) and (2). Most of the above methods in terms of the electric and/or magnetic field strength have counterparts in terms of the electric and/or magnetic flux density as well as in terms of combinations of field strengths and flux densities. We will not discuss those options since they would require the use of face elements along interfaces, which is less efficient than the use of edge elements along interfaces.

3 Compatibility relations

Compatibility relations, that were introduced first by Love [2], are properties of a field that are direct consequences of the field equations and that must be satisfied for those field equations to have a solution (see Mur[3] for the electromagnetics case). For the electromagnetic field equations we have the interior compatibility relations

$$\nabla \cdot (\partial_t \epsilon \cdot E + \sigma \cdot E) = -\nabla \cdot J^{\text{imp}},$$  

(6)

$$\partial_t \nabla \cdot (\mu \cdot H) = -\nabla \cdot K^{\text{imp}},$$  

(7)

the interface compatibility relations

$$\nu \cdot (\partial_t \epsilon \cdot E + \sigma \cdot E) + \nu \cdot J^{\text{imp}} \text{ continuous across the interface } \mathcal{I},$$  

(8)

$$\nu \cdot \partial_t (\mu \cdot H) + \nu \cdot K^{\text{imp}} \text{ continuous across the interface } \mathcal{I},$$  

(9)

and the outer boundary compatibility relations

$$\nu \cdot (\partial_t \epsilon \cdot E + \sigma \cdot E) = \nu \cdot (\nabla \times H^{\text{ext}} - J^{\text{imp}}) \text{ on } \partial \mathcal{D}_H,$$  

(10)
Note that the compatibility relations are all related to the divergence of the electric and the magnetic field strengths. They follow directly from (1) - (4) but may not follow from their discretized counterparts because of which they have to be taken into account explicitly in a finite-element code. The reader will have no difficulty in obtaining compatibility relations for time-harmonic and for static and stationary fields from the above ones. A mathematical analysis of the importance of compatibility relations was recently presented by Jiang et al. [4] for the case of homogeneous media.

In the case of static and stationary fields the above local compatibility conditions may have to be supplemented with global compatibility conditions for prescribing integral quantities, such as potential differences between electrodes, see Lager[5].

4 Expansion functions

In our discussion of the choice of expansion functions we will confine ourselves to the use of linear elements on tetrahedra. The reasoning we use for linear expansion functions is easily generalized to higher order expansion functions. The choice of tetrahedral elements is made for topological reasons. For the efficient expansion of electromagnetic fields essentially two types of consistently linear expansion functions are available, viz. nodal expansion functions and edge expansion functions. Mur [6] discussed the relative merits of each of these functions.

Nodal expansion functions on a tetrahedron are defined through

\[ W_{i,j}^{(N)}(x) = \phi_i(x) i_j \quad \text{with} \quad (i = 0, ..., 3, j = 1, 2, 3), \]  

where \( i_j \) are the base vectors with respect to the (background) Cartesian reference frame and where \( \phi_i(x) \) are the barycentric coordinates of the tetrahedron. Nodal expansion functions ensure the continuity of all field components and they can therefore only be used in regions where all field components are continuously differentiable functions of the spatial variables. In case of discontinuities in the material properties, discontinuities in the normal components of the field across those discontinuities must be expected. Edge expansion functions have been designed to cope with this difficulty.

Edge expansion functions on a tetrahedron are defined through

\[ W_{i,j}^{(E)}(x) = a_{i,j} \phi_i(x) \nabla \phi_j(x), \quad \text{with} \quad (i, j = 0, ..., 3, i \neq j) \]  

where \( a_{i,j} = |x_i - x_j| \) denotes the length of the edge joining the vertices \( x_i \) and \( x_j \) of the tetrahedron. They ensure the continuity of tangential field components while leaving the normal components free to jump. Edge expansion functions have the additional advantage that they reduce the errors that are made on the expansion of the field near reentrant corners.
in the outer boundary of the domain of computation. The combined use of nodal and edge elements requires the use of both element types in the same tetrahedron and the expansion of, for instance, the electric field strength in a tetrahedron can now be written as

$$E(x, t) = \sum_{i=0}^{3} \sum_{j} e_{i,j}(t) W_{i,j}^{(N,E)}(x),$$

where \(\{e_{i,j}(t)\}\) denotes the local set of unknown time-dependent expansion coefficients and where \(j\) depends on the type of expansion used. A detailed discussion of the choice between nodal and edge expansion functions was given by Mur[7].

5 On vector potentials

Since the use of vector potentials (see for instance Preis et al. [10]) is, especially for low-frequency and static or stationary problems, a very popular alternative to our approach we will make a comparison between these two methods as regards their efficiency.

5.1 A comparison of accuracies with field-based methods

In our analysis we will first assume that both finite-element methods use linear nodal expansion functions based on tetrahedra and that the method for computing the electric and/or magnetic field strength directly uses linear edge elements along interfaces. Under these conditions it can be shown that both methods use about the same number of unknowns and our comparison of the efficiencies reduces to a comparison of the accuracies that are obtained.

As regards the accuracy that is obtained when using a finite-element method based on linear expansion functions it is easily shown that the local approximation error is \(O(h^2)\), where \(h\) denotes the largest value of an edge of a tetrahedron locally. Consequently, both methods yield the same local error in the unknown they solve for and, although it is in general not possible to generalize local errors to global errors, a similar behavior can be expected for global errors. For complicated configurations involving sharp corners and/or large contrasts in the medium properties the global accuracy usually deteriorates as compared with the local accuracy, but then both methods suffer from this. The difference between methods using a vector potential and methods that compute the electric and/or magnetic field strength directly becomes clear when one realizes that the user will usually require results for physical quantities, such as the electric and/or magnetic field strength rather than potentials. Unfortunately, when using potentials, those physical quantities can only be obtained by carrying out a numerical differentiation on the potentials. This causes a loss of accuracy because of
which the local approximation error of physical quantities computed from vector potentials is $O(h)$, which is one order less than the local error in the results obtained by a direct computation of the physical quantities with the same computational effort.

5.2 The use of divergence free elements

In some methods using vector potentials a special type of edge element, that is free of divergence, is employed in an attempt to increase the efficiency of the method, see Takahashi[8], and to avoid spurious solutions. In our discussion of those mixed elements we will, for consistency reasons, confine our attention to the linear Whitney 1 elements on tetrahedra that are of this type and that were extensively used by Bossavit [9]. For the relation of these elements to the edge elements used above see Mur [6].

Contrary to what is thought by some, Whitney elements, or edge elements in general, cannot be used to enforce the elimination of spurious solutions from a solution. This was shown by Mur [6] for methods that compute the electric field strength directly, a similar example for methods using potentials can be constructed easily.

Regarding the increase of the efficiency of a potential based method by using Whitney 1 elements the situation is more promising. Replacing linear nodal expansions by linear Whitney 1 elements on tetrahedra causes a slight increase in the number of unknowns (for hexahedral elements the number of unknowns remains the same) but a significant decrease in the connectivity of the system matrices with a corresponding increase in efficiency of the method. Although the asymptotic behavior of the local error in the potential reduces to $O(h)$ when using Whitney 1 elements, the local error in the field values remains unchanged at $O(h)$ which surprising result is a consequence of the fact that no differentiations are carried out in the direction of the governing edges of the Whitney elements. In summary it can be concluded that for computing field values direct methods are still preferable to potential based methods, even when Whitney elements are used in the potential based method. Whitney elements have computational advantages within the context of vector potentials only.

6 The formulation

The formulation of the finite-element method employed in the present paper is in terms of the electric field strength and is based on the weighted residual solution of the equations obtained by substituting (14) into (5) together with the compatibility relations (6), (8) and (10). The remaining compatibility relations are not relevant when (5) is used since they all apply to the magnetic field strength which was eliminated from the formulation. For the resulting system of coupled ordinary differential equations in terms
of the expansion coefficients we employ an implicit formulation, solving the
system of equations at each time step. Time harmonic problems are solved
in the time domain by using a transient to time-harmonic.

7 Numerical results

As an example we compute the field distribution in a cubic numerical phan-
tom of a human head as discussed in COST 244. The field is caused by
the antenna of cordless telephone operating at 900MHz. The phantom (see
Fig. 2) consists of a cubic region of 19x19x19cm for modeling brain tissue
(\(\varepsilon_r = 43, \sigma = 0.83\text{S/m}, \mu_r = 1, \rho =1050 \text{ kg/m}^3\)) surrounded by a 5mm
thick layer for modeling the scull (\(\varepsilon_r = 17, \sigma = 0.25\text{S/m}, \mu_r = 1, \rho =1200 \text{ kg/m}^3\)). The antenna is a 12.6cm long dipole, having its center at \(\mathbf{x}_d = (0.115,0.0,0.0)\), i.e. at 1.5cm from the scull and having a thickness of 2.5mm
and being oriented in the z-direction. The antenna wire is assumed to be

\[
\begin{align*}
E_T &= 0 \\
H_T &= 0
\end{align*}
\]

Figure 2: The numerical phantom, one quadrant.
conditions that model outgoing plane waves at the remaining four faces of the domain of computation. The meshing was chosen such that a dense mesh was generated in the immediate vicinity of the antenna wire for facilitating the almost singular behavior of the field near the antenna. A much coarser mesh was chosen at larger distances of the antenna. The total number of unknowns is 114085, 3082 of them being prescribed explicitly. A contour plot of the normalized specific absorption rate (SAR) in the cross-section \((0 \leq x \leq 0.1, y = 0.0, 0 \leq z \leq 0.1)\) is given in Fig.3. With respect to this figure the antenna is oriented vertically and located just left of the domain depicted.

![Contour plot of normalized SAR](image)

Figure 3: The normalized SAR in the plane \(y = 0\) in dB.

For solving the time-harmonic problem, a transient to time-harmonic was made that required a total of 240 time steps. At each step the implicit system of equations was solved iteratively, requiring 8 iterations per solution in the early stages of the process of solution and only 1 iteration per solution during the final cycle of the solution process, the average being 4.3 iterations per time step. This low number was achieved by using ICCG preconditioning and by predicting new starting vectors for the iteration process from previous solutions. Note that a time-harmonic formulation of the problem would have required the solution of a complex system of equations of the same dimension, but with a much lower condition. Because of this an iterative solution of the latter system of equations would have been much
more time consuming, a direct solution being prohibitively expensive.

8 Conclusions

We have discussed the finite-element modeling of electromagnetic fields directly in terms of the electric and/or magnetic field strength. Emphasis was given to the use of the compatibility relations for eliminating spurious solutions and to the combined use of edge and nodal element for obtaining an optimum in the computational efficiency of the resulting codes. A numerical example was given demonstrating the validity and the efficiency of our approach.

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References


