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Free vibration analysis of rectangular plates resting on a two-parameter elastic medium

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Abstract

The free vibration analysis of plates resting on a two-parameter elastic medium is studied through a finite element procedure. The associated shape functions are derived from the solution of a beam resting on the two-parameter medium. A parametric study is conducted to identify the influence of various factors on the mode shapes and natural frequencies.

1 Introduction

The study of free vibrations of plates resting on soils, is performed to ascertain that displacements and stresses induced by cyclic loads, such as random loads from earthquakes, in machine foundations, on highway and airport pavements are within design specifications. To render the analysis simpler various assumptions are made among which the most significant is that the supporting soil is a Winkler medium. Studies by a number of researchers (Cheung and Zienkiewicz[1]; Celep[2]; Alvappillai and Zaman[3]; Zaman et al.[4]) have focused on obtaining the desired governing differential equation (GDE) for the problem. However, in the vibration of plates instead of the external forces there are inertial body forces and the GDE has to be modified accordingly. In addition, a complication arises, when the plates do not have their two opposite edges supported, in which case trial displacement functions are utilized (Agrawal[5]).

In an attempt to refine the Winkler model the "two-parameter" model has been suggested which permitted discretization of the problem into small elements when using finite element-type solution technique and deriving stiffness and mass matrices for each element (Alvappillai and Zaman[3]). A number of two parameter models are available in the literature; among these the models proposed by Filonenko-Borodich[6], Hetenyi[7], Pasternak[8] and Vlasov-

Leontev [9] are some of the particularly notable ones (Alvappillai and Zaman[3]). In these models, except the Vlasov model, the shear interaction between the individual membrane elements, elastic plate elements or elastic layers are capable of undergoing pure shear deformation. The Vlasov model, on the other hand, achieves the shear interaction by imposing restrictions on the distribution of displacements and stresses in the elastic halfspace (Kerr[10]).

In the present study, Bogner's[11] approach of using a cubic Hermite polynomial is used to evaluate the desired stiffness and mass matrices for the thin plate element. However, instead of using the Hermite polynomial to represent displacements, the shape functions derived by Low[12] are employed. Low's functions were derived from the actual deflected shape of a beam resting on the Vlazov and Leontiev's elastic two-parameter medium. The stiffness and mass matrices are then used to develop a computer algorithm based on the finite element method for the free vibration analysis of plates. A parametric study is conducted to investigate the effect of various parameters on the natural frequencies and mode shapes of plates.

2 Finite Element Algorithm

The free vibration analysis of a plate resting on a soil medium using finite element method (FEM) requires formulation of stiffness and mass matrices of the plate-soil system. Bogner[11] used a cubic Hermite polynomial to model transverse displacement of a beam element (Desai and Abel[13]). In the present study, the basic function for a beam is used to model the transverse displacement of a plate. The plate is discretized by simple elements (Fig. 1) and having the following degrees of freedom at each node: vertical displacement, w_i; rotation with respect to x-axis, $\theta_{xi} = \frac{\partial w}{\partial x}$; rotation with respect to y-axis, $\theta_{yi} = \frac{\partial w}{\partial y}$; and torsion $\theta_{xyi} = \frac{\partial^2 w}{\partial x\,\partial y}$, where i represents the node number.

This results in a total of sixteen degrees-of-freedom for a quadrilateral element having four corner nodes. The Hermite polynomials that can be used as shape functions to represent the displacement characteristics of a one-dimensional beam in x and y directions can be expressesd as:

$$\left.\begin{aligned} NX1 &= 1-3s^2+2s^3 \\ NX2 &= s^2(3-2s) \\ NX3 &= as(s-1)^2 \\ NX4 &= as^2(s-1) \end{aligned}\right\} \quad \begin{aligned} &s = x/a \\ &0 \le s \le 1 \\ &\frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial s} \end{aligned} \qquad (1)$$

$$\left.\begin{aligned} NY1 &= 1-3t^2+2t^3 \\ NY2 &= bt(3-2t) \\ NY3 &= t^2(t-1)^2 \\ NY4 &= bt^2(t-1) \end{aligned}\right\} \quad \begin{aligned} t &= y/b \\ 0 \leq\ & t \leq 1 \\ \frac{\partial}{\partial y} &= \frac{1}{b}\frac{\partial}{\partial t} \end{aligned} \qquad (2)$$

where, a and b are the linear dimensions of a plate in x and y directions, respectively.

By using these shape functions, the displacement field for a plate bending element can be written as (Desai and Abel[13])

$$\begin{aligned} w(x,y) = {} & N1w_1 + N2\,\theta_{x1} + N3\,\theta_{y1} + N4\theta_{xy1} \\ & + N5w_2 + N6\,\theta_{x2} + N7\,\theta_{y2} + N8\theta_{xy2} \\ & + N9w_3 + N10\,\theta_{x3} + N11\,\theta_{y3} + N12\,\theta_{xy3} \\ & + N13w_4 + N14\,\theta_{xy2} + N15\,\theta_{xy3} + N16\,\theta_{xy4} \end{aligned} \qquad (3)$$

where N1, N2,....N16 are functions of NX1, NX2,...NY4.

In the present study, instead of using NX1, NX2,... NY4 that are applicable to a purely beam bending problem, a different set of shape functions are utilized that reflect the actual response of a beam resting on a two-parameter elastic medium and subjected to bending.

To this end, consider the GDE for bending of a beam supported on a two-parameter elastic medium:

$$EI\frac{d^4w(x)}{dx^4} - k_1\frac{d^2w(x)}{dx^2} + k_2w(x) = 0 \qquad (4)$$

where EI is the flexural rigidity, w(x) the deflection at a point x, E the modulus of elasticity (beam), and I the moment of inertia of the beam cross section.

The parameter k_1, called the moment foundation modulus, is a measure of the transmissibility of an applied force to the neighboring elements. Also, the parameter k_2, called the Winkler or subgrade modulus, provides a measure of the tendency of soil medium to deform under an applied compressive stress.

As shown by Selvadurai[14] and Low[12], by employing the method of initial parameters, the general solution of Eq. (4) can be written as

$$w(x) = A_1w_0 + A_2\theta_0 + A_3M_0 + A_4N_0 \qquad (5)$$

where, $A_1 = \phi_2 - \frac{\mu^2-\alpha^2}{2\mu\alpha}$; $A_2 = \frac{1}{2\lambda}\left[\frac{\phi_1}{\mu} + \frac{\phi_3}{\alpha}\right]$; $A_3 = -\frac{\phi_4}{2\mu\alpha\lambda^2 EI}$; $A_4 = \frac{-1}{4\lambda^3 EI}\left[\frac{\phi_1}{\mu} - \frac{\phi_3}{\alpha}\right]$;

$\phi_1 = \cos(\alpha\lambda x)\sinh(\mu\lambda x)$; $\phi_2 = \cos(\alpha\lambda x)\cosh(\mu\lambda x)$; $\phi_3 = \sin(\alpha\lambda x)\cosh(\mu\lambda x)$; $\phi_4 = \sin(\alpha\lambda x)\sinh(\mu\lambda x)$; $\lambda = \left[\frac{k_2}{4EI}\right]^{1/4}$; $\mu = \left[1+\frac{k_1}{k_2}\lambda^2\right]^{1/2}$ for $0 \leq \mu \leq 1$; $\alpha = \left[1-\frac{k_1}{k_2}\lambda^2\right]^{1/2}$ for $0 \leq \alpha \leq 1$; $k_1 = \frac{E_o\, b[s\,\sinh(2\gamma H/s) - 2\gamma H]}{8\gamma(1+v_o)\sinh^2(\gamma H/s)}$ and

$$k_2 = \frac{E_o\, b\gamma[s\,\sinh(2\gamma H/s) + 2\gamma H]}{4s^2(1-v_o^2)\sinh^2(\gamma H/s)}.$$

Here w_o, θ_0, M_o, and N_0 are the deflection, slope, bending moment and transverse shear force, respectively, at the left end of the beam (at x=o). Also, H is the thickness of the soil layer and s is the half-width of the beam. As shown by Low[12], the shape functions for the beam element can be expressed as

$$N_1 = A_1 + A_3K_{21} + A_4K_{11} \quad \textbf{(6a)}$$

$$N_2 = A_3K_{23} + A_4K_{13} \quad \textbf{(6b)}$$

$$N_3 = A_2 + A_3K_{22} + A_4K_{12} \quad \textbf{(6c)}$$

$$N_4 = A_3K_{24} + A_4K_{14} \quad \textbf{(6d)}$$

These shape functions, called "basic functions", are used in this study to derive the components in Eq. (3) for the displacement functions of a rectangular plate element. Details of the formulation and definition of terms are given by Agrawal[5].

3 Stiffness Matrices

By considering the GDE for a plate resting on a two-parameter elastic medium, the total strain energy of the element and its variation, the bending stiffness matrices for the plate element $[k_p]$ can be expressed as
where $[B]_p$ is a transformation matrix, $[C_p]$ is the constitutive relation matrix, h

$$[k]_p = h\int_o^x\int_o^y [B]_p^T[C_b][B]_p\; a\; b\; dxdy \quad (7)$$

where $[B]_p$ is a transformation matrix, $[C_p]$ is the constitutive relation matrix, h is the plate thickness and T represents transpose. Likewise, the stiffness matrix of the two-parameter foundation soil medium $[k]_f$ can be expressed as:

$$[k]_f = [k]_1 + [k]_2 \tag{8}$$

where,

$$[k]_1 = \int_o^x \int_o^y [N]^T \; k_1 \; [N] \; a \; b \; dxdy \tag{9a}$$

$$[k]_2 = 2\int_o^x \int_o^y [B]^T \; k_2 \; [B] \; a \; b \; dxdy \tag{9b}$$

$$[B] = \begin{bmatrix} \frac{\partial N1}{\partial x}, & \frac{\partial N2}{\partial x}, & \ldots\ldots, & \frac{\partial N16}{\partial x} \\ \frac{\partial N1}{\partial y}, & \frac{\partial N2}{\partial y}, & \ldots\ldots, & \frac{\partial N16}{\partial y} \end{bmatrix} \tag{9c}$$

and other terms are given by Agrawal[5].

4 Modeling Mass of the Plate

The mass of the foundation soil is neglected here, only the plate mass is used in determining the element mass matrix. The consistent mass formulation is used. Accordingly, the element mass matrix is given as

$$[m] = \int_v \rho A [N]^T [N] \; dv \tag{10}$$

where ρ is the mass density of the plate material and V is the volume of the plate element.

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5 Modal Analysis

By applying the principle of virtual work and the variational principle, the governing differential equation for a thin plate resting on two-parameter elastic medium can be written in the form:

$$[K]\ \{W\} + [M]\ \{\ddot{\omega}\} = 0 \tag{11}$$

where [K] and [M] are the system stiffness and system mass matrices, respectively, obtained by assembling the element stiffness and mass matrices.

The generalized eigenproblem for this case can be obtained as

$$[K]\ \{\phi\} = \{w^2\}\ [M]\ \{\phi\} \tag{12}$$

from which the eigenvector or the mode shapes ϕ and the corresponding eigenvalues ω or the natural frequencies of the plate-elastic support system are determined.

6 Numerical Results

In most numerical analyses only the first few natural frequencies of vibration and the corresponding mode shapes are important from an engineering design point of view. Therefore, natural frequencies and modes shapes are determined here only for the first five modes of free vibration of a plate on a two-parameter elastic medium. The first five frequencies represent the lowest five frequencies of the plate. An extensive parametric study was conducted to investigate the influence of various factors. Only selected results are presented here due to space limitation. Figure 2 shows the variation of frequency parameter (Ω) for the first five modes of a plate resting on an elastic medium with all four edges simply supported. The horizontal axis represents the flexibility index that include variations of geometric properties of the plate, (namely, length, width, thickness and flexural rigidity), as well as the modulus of elasticity (E) and Poisson's ratio (μ) of the elastic medium supporting the plate (see Agrawal[5] for definition of Ω and γ). Figure 3 shows similar results for a plate resting on an elastic medium but clamped at all four edges. It is observed that frequency parameters (Ω) for a clamped plate are consistently higher than that of a simply supported plate, as expected. However, values of the flexibility index are lower for all modes of vibration. A decrease in the value of the flexibility index can be attributed to constraining of more degress of freedom in the clamped plate as compared to the simply supported plate. Constraining the supports also results in increased stiffness of the clamped plate which causes an increase in the frequency parameters of the clamped plates. An increase in the frequency

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parameters at higher modes is also expected.

7 Effect of Loss of Support

Due to the vibration of the plates, the loss of contact between plate and supporting medium can occur, especially in the case of a small super-structure supported by a plate resting on an elastic medium. To simulate this effect, partially supported plates are analyzed and the resulting mode shapes are compared with those for the fully unsupported plates.

The results for the mode shapes are compared for various values of β (ratio of the supported length to the total length of plate) with the corresponding mode shapes of a fully supported along with supports at all four edges. The first mode of vibration of a fully unsupported plate ($\beta = 0.0$) in Fig 4 is compared with a partially supported plate ($\beta = 0.25$) in Fig 5. Because of the elastic medium support at quarter length of the plate near the right edge (Fig. 5), the plate is stiffer than a fully unsupported plate which causes a decrease in the normalized mode shape amplitude near the right edge of the partially supported plate. Contours of normalized mode shape amplitude in Fig. 5 show a shift toward the left edge because of the increased stiffness of the plate near the right edge.

Conclusions

This study has demonstrated that the increase in the flexibility index of the plate decreases its natural frequency and that such an increase is obtained by increasing the length to width ratio of the plate. Also, loss of support has profound effect on natural frequency and mode shapes.

References

1. Cheung, Y.K. and Zienkiewicz, O.C. Plates and tanks on elastic foundations-an application of finite element method, *International Journal of Solids and Structures*, 1965, Vol. 1, No. 4, pp. 451-461.

2. Celep, Z. Circular plates on tensionless Winkler foundation, *Journal of Engineering Mechanics*, 1988, Vol. 114, No. 10, pp. 1723-1739.

3. Alvappillai, A. and Zaman, M.M. Dynamic response of foundations of two-parameter media, *Proc. Second International Conference on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics*, Vol. 1, pp. 679-684, St. Louis, MO. 1991.

4. Zaman, M.M., Alvappillai, A., and Taheri, M.R., Dynamic analysis of concrete pavements resting on a two-parameter medium, *International Journal for Numerical Methods in Engieering*, 1993, Vol. 36, pp. 1465-1486.

5. Agrawal, A. *Free Vibration Analysis of Rectangular Plates Resting on a Two-Parameter Elastic Medium*, M.S. thesis, the University of Oklahoma, OK. 1993.

6. Filonenko-Bordich, M.M. Some approximate theories of the elastic foundation, (in Russian), *Uchenyie Zapiski Moskovskogo Gosudarstuennogo Universiteta Mechanika*, 1940, No. 46, pp. 3-18.

7. Hetenyi, M. *Beams on Elastic Foundation*, University of Michigan Press, Ann Arbor, Mich., 1946.

8. Pasternak, P.L., On a new method of analysis of an elastic foundation by means of two foundation constants, (in Russian), *Gosudarstvennoe Izdatelstvo Literaturi po Stroitielstvu i Arkhitekture*, Moscow, 1954.

9. Vlasov, V.Z. and Leontev, N.N., *Beams, Plates and Shells on Elastic Foundation*, Published for NASA and NSF by the Israel Program for Scientific Translation, 1966.

10. Kerr, A.D. Elastic and viscoelastic foundation models, *Journal of Applied Mechanics*, Transaction of the ASME, pp. 491-498, 1964.

11. Bogner, F.K., Fox, R.L., and Schmit, L.A. The generation of inter-element-compatible stiffness and mass matrices by the use of interpolation formulas, *Proceedings of the Conference on Matrix Methods in Structural Mechanics*, AFFKL-TR-66-80, pp. 397-440, Wright-Patterson Air Force Base, OH, 1966.

12. Low, W.C. *Analysis of Free Vibration of Beams and Plane Frame Resting on the Vlasov's Two-Parameter Elastic Medium*, M.S. thesis, University of Oklahoma, OK, 1990.

13. Desai, C.P. and Abel J.F. *Introduction to the Finite Element Method*, Von Nostrand Reinhold, New York, NY, 1972.

14. Selvadurai, A.P.S. *Elastic Analysis of Soil-Foundation Interaction*, Elsevier Scientific Publishing Company, Amsterdam, Holland, 1979.

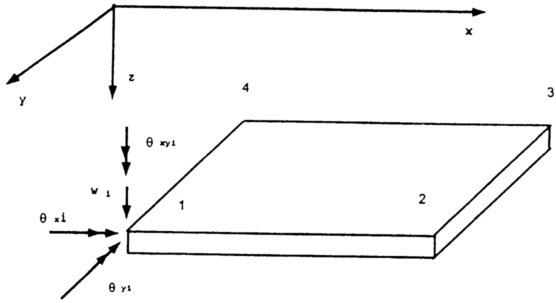

Figure 1: A Quadrilateral Plate Element With Four Degress of Freedom

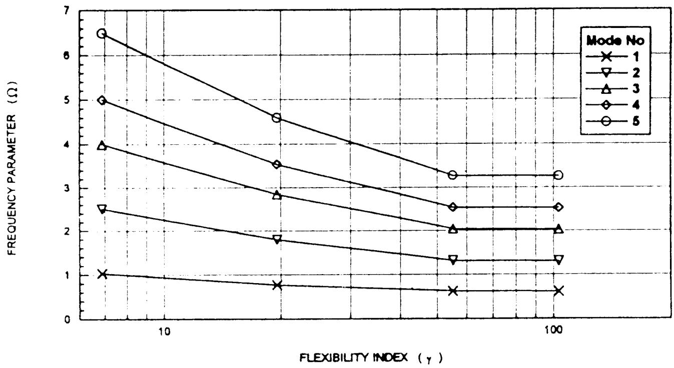

Figure 2: Variation of Flexibility Index Vs Frequency Parameter for a Plate Simply Supported at all Four Edges [$\mu = 0.45$; $\phi = 1.0$]

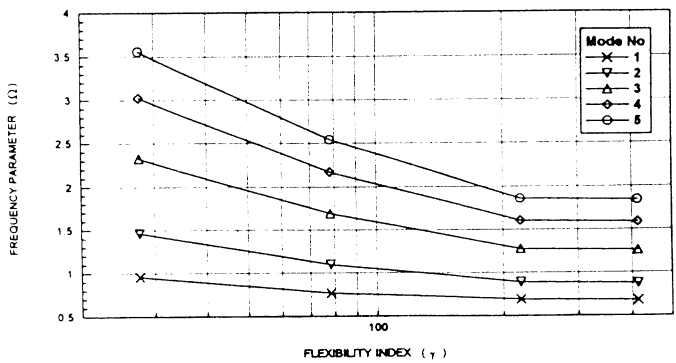

Figure 3: Variation of Flexibility Index Vs Frequency Parameter for a Plate Simply Supported at all Four Edges [$\mu = 0.45$; $\phi = 2.0$]

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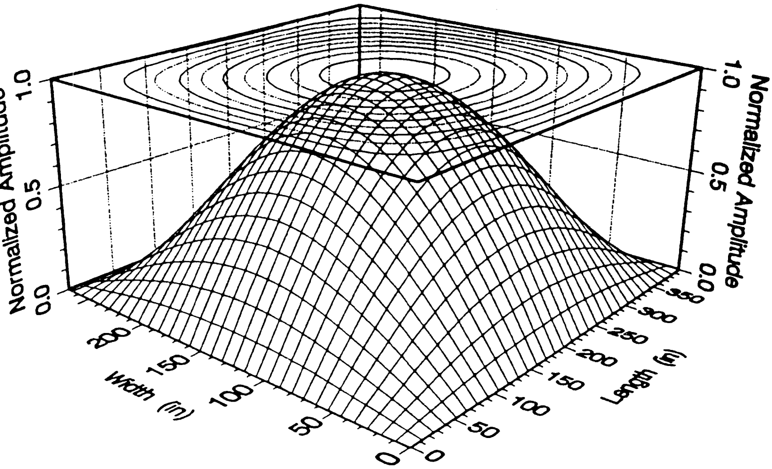

Figure 4: Mode Shape of Free Vibration of a plate Simply supported at all Four Edges : 1st Mode [μ = .45; ϕ = 1.5; β = 0.0]

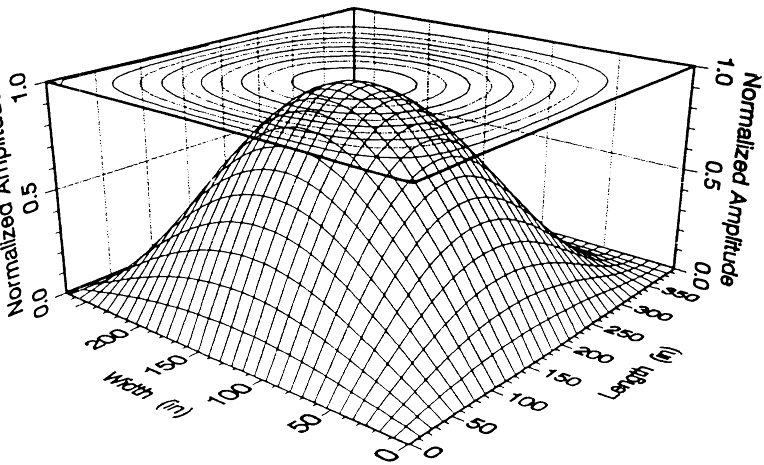

Figure 5: Mode Shape of Free Vibration of a Plate Simply Supported at all Four Edges : [μ = .45; ϕ = 1.5; β = 0.25]