Dynamic response of underground structures in soils with variable mechanical properties

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Abstract

In this work, we develop a formulation based on the Boundary Element Method (BEM) for evaluating the dynamic response of underground structures excited by seismically-induced, steady-state, horizontally polarized shear (SH) waves. The surrounding geological medium is viewed as an elastic continuum exhibiting large randomness in its mechanical properties. Suitable Green's functions are proposed and subsequently used within the context the BEM formulation. More specifically, the methodology developed herein employs a series expansion for the proposed Green's functions, where the basis functions are orthogonal polynomials of a random argument. These functions are incorporated in a BEM formulation, which is then used for the solution of problems of engineering interest. The presented approach departs from earlier BEM derivations based on perturbations, which were only valid for "small" amounts of randomness in the elastic continuum.

1 Introduction

For many years in engineering practice, dynamic response analysis of underground structures was rarely, if ever, performed due to a widespread belief in the engineering community that these structures were not susceptible to damage from earthquake excitations. It was only after recent failures of important underground structures during high intensity earthquakes (a notable example being the Hyogo-Ken-Nanbu earthquake of January 17, 1995, near the city of Kobe, Japan) that the fallacy of such a belief became evident [1]. Thus, there is growing need in earthquake engineering practice for the development of
numerical methods that will address the efficient dynamic analysis of key underground structures that comprise part of the basic infrastructure of a region. Many of these buried structures serve as lifelines (e.g., tunnels, pipelines), and a basic problem with their design using conventional FEM/BEM software packages stems from lack of information about the composition and mechanical properties of the geological deposits that surround the structure. This information is usually not available with sufficient accuracy for a number of reasons, such as sheer difficulty and/or economic cost associated with measurements and testing. There is, however, an alternative proposition, whereby the geological medium is viewed as continuous, but with stochastic material properties. Generally speaking, irregular changes seen against the background of a piecewise homogeneous (layered) deposit give rise to uncertainty, which is then manifested in the medium’s response. Thus, representation of the geological medium as being random is an attractive idea, in the sense that it bypasses the need for extensive field-testing, but still allows for rational approximations to be made for the stress/strain states that ultimately develop in the buried structure.

Problems involving random media are governed by stochastic differential equations, which, by proper decomposition of the differential operator into deterministic plus random parts, can be recast as random integral equations. Approximate solutions can then be generated by applying the expectation operator to the random integral equation and then using various closure approximations, by using perturbations, or by other techniques [2]. One of the first works on the propagation of elastic waves in a random, continuous medium was by Karal and Keller [3]. An extension of their work to multi-layered systems was proposed by Kotulski [4]. From a numerical viewpoint, two broad classes for evaluating the dynamic response of continuous media can be distinguished, namely simulation techniques [5] and approximate methods. A detailed review of the latter category of methods for waves in continuous as well as in discrete stochastic media and for wave scattering by stochastic surfaces can be found in Sobczyk [6]. Finally, in an effort to overcome the limitation of small parameter uncertainty, series expansions of the random response of a system in polynomials that are orthogonal with respect to the expectation have appeared for the FEM [7]. An analogous approach was proposed by Manolis and Karakostas [8] concerning the development of Green’s functions for the case of SH waves propagating in a geological medium that exhibits large randomness in its mechanical properties.

This work uses the ‘polynomial chaos expansion’ methodology proposed in [8] as a basis for the development of a stochastic BEM formulation, suitable for the computation of the dynamic response of underground structures in random media subjected to steady-state, SH waves under anti-plane strain conditions.
2 Polynomial chaos expansion methodology

A summary of the theoretical background which serves as a basis for this work is presented in this section. The detailed theoretical development for the 'polynomial chaos expansion' can be found in [8].

Time harmonic waves that propagate in a random continuum under anti-plane strain conditions are governed by the Helmholtz’s equation

\[ \nabla^2 u(x, y) + k^2(\gamma)u(x, y) = f(x, y) \]  \hspace{1cm} (1)

where \( u \) is the displacement component in the \( z \)-direction and \( k \) is the wave number equal to \( \omega/c(\gamma) \), with \( \omega \) the frequency and \( c \) the wave speed. Furthermore, \( x \) is the position vector restricted to lie on the \( x-y \) plane and \( \gamma \) is a random parameter. Note that factor \( \exp(-i\omega t) \), where \( t \) is the time, is implied and that \( \nabla^2 \) is Laplace’s operator.

For the deterministic medium, eqn (1) in cylindrical coordinates becomes

\[ \frac{d^2 u(r)}{dr^2} + \frac{1}{r} \frac{du(r)}{dr} + k_0^2 u(r) = \delta(r) \]  \hspace{1cm} (2)

with \( k_0 \) the mean wave number.

Stochasticity results due to randomness in the wave number, which is defined in terms of a mean plus a fluctuating component as

\[ k(\gamma) = k_0 + k_1 \gamma \]  \hspace{1cm} (3)

In the above, \( k_1 \) is a deterministic coefficient and \( \gamma \) is a random variable with zero mean and unit variance \( \sigma_\gamma^2 \). Next, we expand both fundamental solution \( u(r, \gamma) \) and forcing function \( f(r, \gamma) \) as a series in terms of an orthogonal set of polynomials in \( \gamma \). This implies a separation of variables, since the orthogonal polynomials \( P_n \) are weighted by spatially dependent coefficients \( U_n \) (similarly, \( F_n \) are the forcing function coefficients) in the form

\[ u(r, \gamma) = \sum_{n=0}^{N-1} P_n(\gamma) U_n(r) \]  \hspace{1cm} (4)

where \( N \) is the order of approximation in the random space and

\[ P_n(\gamma) = 2^{-n/2} \pi^{-1/4} (n!)^{-1/2} H_n(\gamma) \]  \hspace{1cm} (5)

In the above equation, \( H_n \) are the Hermite polynomials of order \( n \).
The next step is substitution of the expansions given by eqns (4) into the governing equation of motion, which after some algebra yields the following coupled system of differential equations, which governs coefficients $U_n(r)$:

$$
\begin{align*}
&h_{m,n}F_n = h_{m,n} \nabla^2 U_n + \\
&\left[ k_i^2/4 \right] h_{m,n+2} + k_0 k_1 h_{m,n+1} + \left[ k_i^2 + k_i^2 (n+1/2) \right] h_{m,n} + 2n k_0 k_1 h_{m,n+1} + n(n-1) k_i^2 h_{m,n+1} \right] U_n
\end{align*}
$$

(6)

In the above, the 'weights' are $h_{m,n} = \langle H_m(\gamma) H_n(\gamma) \rangle = \sqrt{\pi} \ 2^n (n!) \delta_{nm}$. By carrying out the expansion (e.g., for five terms with $n = 0, 1, \ldots, 4$), we obtain the following matrix differential equation:

$$
\frac{d^2}{dr^2} \{U\} + \frac{1}{r} \frac{d}{dr} \{U\} + \mathbf{K} \{U\} = \{F\}
$$

(7)

With respect to the above equation, we note that system matrix $[K]$ is non-symmetric and has furthermore been truncated. In order to uncouple eqns (7), the system matrix must be diagonalized by using its eigenvalues $[\lambda_1, \lambda_2, \ldots, \lambda_n]$ and the corresponding eigenvector matrix $[\Phi] = [\{\phi\}_1, \{\phi\}_2, \ldots, \{\phi\}_n]$. By defining a new set of variables as

$$
\{V\} = [\Phi]^{-1} \{U\} \quad \text{and} \quad \{B\} = [\Phi]^{-1} \{F\}
$$

(8)

an uncoupled system of Helmholtz equations results as

$$
\frac{d^2}{dr^2} \{V\} + \frac{1}{r} \frac{d}{dr} \{V\} + \mathbf{A} \{V\} = \{B\}
$$

(9)

The first fundamental solution for outgoing waves is simply

$$
V_n(r) = (i/4) H_n^{(1)}(\lambda_n r)
$$

(10)

where $\lambda_n$ is the wave number corresponding to the $n$th term of the expansion. The expression for the derivative of $V_n$, which corresponds to the second fundamental solution for the traction, is similar.

The first two statistical moments of the response can then be computed for any nodal value resulting from the stochastic BEM solution of a boundary-value problem. The mean value for the displacements is given by

$$
m_n(r) = \langle u(r, \gamma) \rangle = \langle H_n(\gamma) U_n(r) \rangle = \langle H_n H_o \rangle U_n =
$$

$$
= h_{0,0} U_0 + h_{0,1} U_1 + \ldots = \sqrt{\pi} U_0(r) + 0 + \ldots = \sqrt{\pi} U_o(r)
$$

(11)
It is obvious that the mean solution is not equal to the deterministic, which is obtained when randomness is absent in the material and the effective wave number of the problem is $k_0$. Thus, the former solution exceeds the latter (in absolute value terms), and the physical explanation is an interference effect caused by continuous scattering of the propagating signal [3]. We note that the amplification factor in question here is not exactly $\sqrt{\pi}$, because the wave number corresponding to $U_0$ does not coincide with $k_0$.

The covariance is computed as

$$\text{cov}_a(r_i,r_j) = \sqrt{\pi} (1 - \sqrt{\pi})^2 U_0(r_i)U_0(r_j) + \sum_{n=1}^{N-1} h_{n,n} U_n(r_i)U_n(r_j)$$

with $h_{0,0} = \sqrt{\pi}$, $h_{1,1} = 2\sqrt{\pi}$, $h_{2,2} = 8\sqrt{\pi}$, $h_{3,3} = 48\sqrt{\pi}$, $h_{4,4} = 384\sqrt{\pi}$, etc.

### 3 Stochastic BEM formulation

Numerical solution of a general boundary-value problem can be accomplished by introducing a set of $N$ boundary integral equations, each one corresponding to the $n^{th}$ order ($n = 0, ..., N-1$) term of the polynomial chaos expansion of the fundamental solutions as

$$c(\xi) u_n(\xi,\omega) = \int_S \left\{ U_n(\xi,\xi,\omega) t_n(\xi,\omega) - T_n(\xi,\xi,\omega) u_n(\xi,\omega) \right\} dS$$

In the above, $\xi$ is the position of the receiver and $\xi$ is that of the source, while $u_n$ and $t_n$ respectively are displacements and tractions at the surfaces of the problem corresponding to the $n^{th}$ order of approximation. Finally, $c$ is the jump term, which depends on the smoothness of the surface at the receiver.

Using BEM formalism, eqn (13) is transformed into a system of linear algebraic equations of the following type:

$$[U_n] [t_n] = [T_n] [u_n]$$

For simplicity, we use the same symbols above as those appearing in the boundary integral equation. In a well-defined problem, half the nodal variables $\{u_n\}$ and $\{t_n\}$ are known from the boundary conditions, while the remaining half are computed from a partitioning and inversion of eqn (14). The complete solution is then obtained as the sum of $N$ terms, as shown in eqn (4) for the displacement. Finally, the response statistics are computed in the manner outlined in section 2.
Table 1: Layout of the key steps of the proposed methodology.

<table>
<thead>
<tr>
<th>Step</th>
<th>Action</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>Read input data</td>
<td>Determine soil mechanical properties, geometry, frequency range $\omega_i$, $i=1, ..., NF$, loading $f_i$, $k_i$ (eqn (3)), etc.</td>
</tr>
<tr>
<td>(2)</td>
<td>For each $\omega_i = 1, ..., NF$ :</td>
<td>Fix the number of expansion terms to be used in eqn (4)</td>
</tr>
<tr>
<td></td>
<td>Set desired expansion level $n=0, ..., N$</td>
<td>From input data, compute system matrix $[K]$ and find eigenvalues $\lambda_n$ and eigenvectors ${\varphi_n}$</td>
</tr>
<tr>
<td></td>
<td>Solve eigenvalue problem: $[K] [\Phi] = [\Lambda] [\Phi]$</td>
<td>Compute ${B} = [\Phi]^{-1} {F}$ (see eqn (8))</td>
</tr>
<tr>
<td></td>
<td>Transform loading to ‘eigenspace’</td>
<td></td>
</tr>
<tr>
<td>(2.1)</td>
<td>For each expansion level $n=0, ..., N$ :</td>
<td>Solve the uncoupled Boundary Value Problem for each expansion level in ‘eigenspace’ using appropriate kernels (eqn (10)). Compute response variables of BVP ${V_n}$ for each expansion level $n$</td>
</tr>
<tr>
<td></td>
<td>Solve BVP : $\frac{d^2}{dr^2} {v} + \frac{1}{r} \frac{d}{dr} {v} + \lambda {v} = {b}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Repeat substep (2.1) for next expansion level $n$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Transform solution of BVP back to original space</td>
<td>Compute response variables of BVP ${U} = [\Phi] {V_n}$ (eqn (8))</td>
</tr>
<tr>
<td></td>
<td>Compute response statistics for $\omega_i$</td>
<td>Compute mean and variance of response for each frequency $\omega_i$ using eqns (11) &amp; (12)</td>
</tr>
<tr>
<td></td>
<td>Repeat Step (2) for the next frequency $\omega_i$</td>
<td></td>
</tr>
<tr>
<td>(3)</td>
<td>Print results</td>
<td>Using the computed response for each frequency $\omega_i$, plot spectral response graphs</td>
</tr>
</tbody>
</table>
For the numerical solution, a special-purpose BEM computer code in Fortran [9] was modified for implementing the proposed polynomial chaos methodology in order to compute the dynamic response of unlined, underground openings subjected to SH waves under anti-plane strain conditions. For the spatial discretization of this 2D problem, three-noded quadratic elements are used in conjunction with an isoparametric approach for the geometry and the traction/displacement fields [9]. Also, since system matrix $[K]$ (eqn (7)) is non-symmetric, an appropriate solution scheme had to be implemented for the computation of its eigenvalues/eigenvectors [10]. To better illustrate the proposed methodology, the key implementation steps are presented in Table 1.

4 Numerical examples

4.1 Circular cavity under SH-wave excitations

Consider a circular tunnel of radius $R=0.381$ m at a large depth from the surface (Figure 1a). The tunnel is subjected to a series of out-of-plane, steady-state SH wave excitations along its entire perimeter. The excitation frequencies considered range from 0.1 to 10.0 Hz, with an increment of 0.1 Hz. The mean values for the material properties of the surrounding soil are $\lambda=270$ MPa, $\mu=180$ MPa and $\rho=2000$ kg/m$^3$, which correspond to a mean wave propagation velocity $c_0 = 300$ m/sec, indicative of soft soil conditions. The perimeter of the tunnel is discretized using 8 three-noded quadratic elements, while test runs with finer discretizations yielded the same results. In Figure 1a, the central node of each element is denoted by a full circle. The problem is solved for the deterministic case (no randomness), as well as for three stochastic cases: (a) $k_l=5\% \, k_0$ (small randomness), (b) $k_l=20\% \, k_0$ (medium randomness) and (c) $k_l=50\% \, k_0$ (large randomness). Obviously, for this problem there is a radial symmetry in the results. In Figure 1b, a spectral plot of the displacement amplitude for the deterministic case is compared with the corresponding mean displacement amplitudes from the three stochastic cases, using a five-term polynomial expansion. In the same figure, the mean $\pm 1$ s.d. curves are also plotted. The computed mean response differs between the deterministic and random cases, due to interference effects caused by continuous scattering of the propagating signal in the random medium, as was previously explained. We also note that for this particular problem, increasing randomness does not affect the response, since the results for all three random cases differ only slightly. This is due to the fact that the wave is immediately radiated in the surrounding soil, and hence the response at a certain node on the perimeter is not greatly affected by the response of its adjacent points. It is also noted here that the perturbation method would yield results simply proportional to the amount of stochasticity in the medium, which clearly would not be correct. From the results of Figure 1b, it is also observed that both mean response and the $\pm 1$ s.d. zone decrease with increasing frequency of excitation, with the slope of the stochastic mean curve being larger than that of the deterministic one throughout the entire frequency range. In Figure 1c, a
similar spectral plot is presented for the displacement phase angle (in degrees) and analogous conclusions can be drawn. The phase angle, from a physical point of view, denotes the time lag of the response with respect to the excitation. Finally, in Figure 1d, the mean displacement amplitudes resulting from the three stochastic cases are compared for a three and five-term expansion of the dependent variables. It is seen that the results obtained are very similar, i.e., even a three-term expansion is sufficient to accurately predict the response for this particular problem.

Figure 1: Circular cavity under SH waves: (a) geometry, (b) displacement amplitude, (c) displacement phase angle, (d) mean displacement amplitude for three versus five term expansions.

4.2 Elliptical cavity under SH-wave excitations

Consider an elliptical tunnel at a large depth from the surface (Figure 2a). The radii of its two axes are $R_1 = 0.381$ m and $R_2 = 2 R_1$. The mean material properties of the surrounding soil, the spatial discretization and the loading
conditions are the same as before. In Figures 2b and 2c, the displacement mean amplitude and phase angle ± 1 s.d. spectral plots for three different amounts of randomness ($k_I = 5\% k_0$, $20\% k_0$ and $50\% k_0$) are presented and compared with the deterministic case. The conclusions drawn are similar to those for the circular tunnel of example 4.1. Finally, in Figure 2d the mean displacements ± 1 s.d of nodes 1 and 5 are presented for the case of large randomness ($k_I = 50\% k_0$) and compared with the deterministic case.

Figure 2: Elliptical cavity under SH waves: (a) geometry, (b) displacement amplitude, (c) displacement phase angle, (d) displacement amplitudes of nodes 1 and 5 for $k_I = 50\% k_0$ compared with the deterministic case.

5 Conclusions

In this work, a stochastic BEM formulation is developed for the solution of soil-structure-interaction problems involving geological media, which exhibit large
randomness in their material properties. The underlying theoretical methodology is based on a series expansion employing Hermite polynomials and satisfactory accuracy can be obtained with as little as three terms. The present method is general enough to be expanded to plane strain and 3D cases, and it is also possible to easily include the liner and other components of an underground opening using conventional techniques.

References


