Time-domain analysis of structures with dampers modelled by fractional derivatives

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Abstract

Two numerical procedures are proposed to solve the equations of motion of multi-degree-of-freedom systems with damping properties expressed by fractional derivatives, representative of the behaviour of several classes of devices used in the field of structural control. The two procedures are modified versions of the central difference integration method, previously introduced for single-degree-of-freedom systems and extended herein to multi-degree-of-freedom cases, and the constant-average-acceleration implicit Newmark method, both implemented for problems governed by fractional derivatives. The generalized formulations of the two integration schemes are presented in terms of recursive equations. The stability and accuracy characteristics are then examined, and an analytical rule for assessing the critical step-size relevant to the central difference method is provided. A demonstrative case study, constituted by a two degree-of-freedom braced frame including fluid viscous dampers, is finally reported.

1 Introduction

The dynamic behaviour of structural systems whose damping properties are defined by fractional derivative (FrD) relationships [1] has generally been analysed in the frequency domain [2], [3], whereas less attention has been paid to time domain computation. A pioneer study conducted on this topic concerned single-degree-of-freedom (SDOF) systems, and was based on the explicit central difference (CD) method suitably modified so as to include FrD calculation [4].
In this paper, the procedure initially used by Koh and Kelly [4], and then applied by Gusella and Terenzi [5] for reproducing the force-velocity response of silicone-based fluid viscous devices, is extended to the integration of the equations of motion of multi-degree-of-freedom (MDOF) systems, for more general applications in the field of passive vibration control. This modified central difference technique – that allows obtaining displacement, velocity and acceleration at the end of the time step, from the corresponding data at the beginning of the same time step without solving a system of equations – is particularly attractive for nonlinear problems, where the governing mass, damping and stiffness matrices could alter continuously. Furthermore, this typically explicit algorithm can be easily programmed, its only disadvantage being the restraints on the step-size imposed by numerical stability conditions.

In order to overcome this restriction, that can become considerably burdensome when large MDOF systems are dealt with, a generalized version of the implicit constant-average-acceleration procedure belonging to the Newmark family ([6], [7], [8]) is then proposed as an alternative approach for time domain FrD analyses. The Riemann-Liouville (LR) algorithm [1] is introduced in the basic schemes of both central differences (CD) and constant-average-acceleration implicit Newmark (IN) methods to perform fractional derivative calculation. Since this combination can notably alter the well-known stability and accuracy properties of the two original methods (established by basic studies concerning linear viscous damped systems [6], [7], [8]), the algorithmic features of the modified versions presented herein are carefully analysed. Concerning stability, this is confirmed to be unconditional for the IN method, whereas a relationship between the critical step-sizes required in the FrD and linear viscous damping problems, is established for CD. Moreover, the accuracy properties are discussed with reference to a demonstrative case study, represented by a 2-DOF framed structure including concentric braces equipped with FV dampers.

2 Integration methods for MDOF systems with FrD damping characteristics

By referring to symbols shown in Figure 1, the equations of motion of a MDOF system with \( n \) degrees of freedom, having both linear viscous and fractional damping components, are:

\[
m_j \ddot{x}_j + c_j \dot{x}_j + c_2 (\ddot{x}_j - \ddot{x}_2) + c_2^* D^q(x_j - x_2) + k_j x_j + k_2 (x_j - x_2) = F_j
\]

\[
m_r \ddot{x}_r + c_r (\ddot{x}_r - \ddot{x}_{r-1}) - c_{r+1} (\ddot{x}_r - \ddot{x}_{r-1}) + c_r^* D^q(x_r - x_{r-1}) - c_{r+1}^* D^q(x_r - x_{r-1}) + k_r (x_r - x_{r-1}) - k_{r+1} (x_r - x_{r-1}) = F_r
\]

\[
m_n \ddot{x}_n + c_n (\ddot{x}_n - \ddot{x}_{n-1}) + c_n^* D^q(x_n - x_{n-1}) + k_n (x_n - x_{n-1}) = F_n
\]
where:

\[ m_1, \ldots, m_n, m_n = \text{masses of the system;} \]

\[ x_1, \ldots, x_n, x_n = \text{lagrangian displacement coordinates (one and two superior dots denote first and second time derivatives);} \]

\[ c_1, \ldots, c_n, c_1^*, \ldots, c_n^* = \text{damping constants relevant to the linear viscous (without asterisk), and the fractional derivative (with asterisk) terms;} \]

\[ D^q = \text{displacement derivative of order } q (<1); \]

\[ k_1, \ldots, k_n = \text{stiffness coefficients.} \]

### 2.1 Application of CD integration procedure

By substituting in (1) the basic central difference relations [6], [7]:

\[ x(t) = x_k; \quad x(t + h) = x_{k+1}; \quad x(t - h) = x_{k-1}; \quad \dot{x}_k = \frac{1}{h}(x_{k+1} - 2x_k + x_{k-1}); \quad \ddot{x}_k = \frac{1}{2h}(x_{k+1} - x_{k-1}) \]

where \( h \) is the integration step-size, the following generalized algorithmic equations are obtained, which define the proposed re-formulation of the CD method for systems with FD damping characteristics:

\[
\begin{align*}
  a_0 m_i(x_{k+1} - 2x_k + x_{k-1}) + a_1 (c_1 + c_2)(x_{k+1} - x_{k-1}) - a_1 c_2 (x_{k+1} - x_{k-1}) + \\
  + a_3(c_1^* + c_2^*) \sum_{j=0}^{k} \varphi_j x_{j+1} - a_1 c_2^* \sum_{j=0}^{k} \varphi_j x_{j+1} + (k_1 + k_2)x_k - k_2 x_{k+1} = F_i
\end{align*}
\]  

\[ 2.1. \]

\[
\begin{align*}
  a_0 m_k(x_{k+1} - 2x_k + x_{k-1}) - a_1 c_1 (x_{k+1} - x_{k-1}) + a_1 (c_1 + c_2)(x_{k+1} - x_{k-1}) + \\
  - a_1 c_2 (x_{k+1} - x_{k-1}) - a_1 c_1^* \sum_{j=0}^{k} \varphi_j x_{j+1} + a_3(c_1^* + c_2^*) \sum_{j=0}^{k} \varphi_j x_{j+1} - a_3 c_1^* \sum_{j=0}^{k} \varphi_j x_{j+1} + \\
  - k_1 x_{k+1} + (k_r + k_r) x_k - k_1 x_{k-1} = F_k
\end{align*}
\]

\[
\begin{align*}
  a_0 m_k(x_{k+1} - 2x_k + x_{k-1}) - a_1 c_1 (x_{k+1} - x_{k-1}) + a_1 (c_1 + c_2)(x_{k+1} - x_{k-1}) + \\
  - a_1 c_2 (x_{k+1} - x_{k-1}) - a_1 c_1^* \sum_{j=0}^{k} \varphi_j x_{j+1} + a_3(c_1^* + c_2^*) \sum_{j=0}^{k} \varphi_j x_{j+1} - k_1 x_{k+1} + k_1 x_{k-1} = F_k
\end{align*}
\]

where:

\[ a_0 = \frac{1}{h^2}; \quad a_1 = \frac{1}{2h}; \quad a_2 = 2a_0; \quad a_3 = \frac{1}{h^q} \]  

\[ (3. a) \]
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\[ \varphi_0 = \frac{1}{\Gamma(2-q)} \left[ (k-1)^{1-q} - k^{1-q} + (1-q)k^{-q} \right] \]  
\[ \varphi_k = \frac{1}{\Gamma(2-q)}; \]  
\[ \varphi_{k-j} = \frac{1}{\Gamma(2-q)} \left[ (j+1)^{1-q} - 2j^{1-q} + (j-1)^{1-q} \right] \quad 1 \leq j \leq k-1 \]  
\[ \frac{d^q x}{dt^q} = t^{-q} N^q \sum_{j=0}^{N} \varphi_j x_j, \quad 0 < q < 1 \]  
\[ N = \text{number of sub-intervals of each time step over which the fractional derivative is evaluated.} \]

2.2 Application of IN integration procedure

To define the modified version of the constant-average-acceleration scheme, the general Newmark time discretization is assumed [6], [7], [8]:

\[ \ddot{x}(t + h) = \dot{x}(t) + [(1-\gamma)\ddot{x}(t) + \gamma\ddot{x}(t + h)]h \quad (4.a) \]

\[ x(t + h) = x(t) + \dot{x}(t)h + \left[ \frac{1}{2} - \beta \right] \ddot{x}(t) + \beta \ddot{x}(t + h) \right] h^2 \quad (4.b) \]

where \( \beta = 1/4 \) and \( \gamma = 1/2 \) are considered.

By substituting (4)’s in the equations of motion (1), and putting:

\[ x(t+h) = x_{k-1}; \]
\[ \dot{x}(t+h) = \dot{x}_{k-1}; \]
\[ \ddot{x}(t+h) = \ddot{x}_{k-1}; \]

the following algorithmic relations are obtained for the IN method:

\[ m \left( \frac{4}{h^2} \dot{x}_{i+1} - \frac{4}{h^2} \dot{x}_i - \frac{4}{h} \ddot{x}_i - \dddot{x}_i \right) + \left[ c_1 + c_2 \left( \frac{2}{h} \dot{x}_{i+1} - \dot{x}_i - \frac{2}{h} \dot{x}_i \right) + \right. \]
\[ - c_2 \left( \frac{2}{h} \dot{x}_{i+1} - \dot{x}_i - \frac{2}{h} \dot{x}_i \right) - \left( \dddot{x}_{i+1} - \dddot{x}_i \right) \frac{1}{h^q} \sum_{j=0}^{N-1} \dot{\varphi}_j x_j, \right. \]
\[ \left. + \frac{1}{h^q} \sum_{j=0}^{N-1} \dot{\varphi}_j x_j + \right) \quad (5) \]
\[ (k_1 + k_2) x_{i+1} - k_2 x_{i+1} = F_{i+1} \]
2.3 Algorithmic properties of the two integration procedures

A considerable increase of computation is caused by introducing the LR algorithm [1] in the CD and IN integration schemes. A check about their numerical properties after this implementation was therefore needed. A numerical investigation was carried out to this aim by considering a series of two, three, and four degree-of-freedom systems characterized by FrDss with q order varying from 0.3 to 0.9 (i.e., the entire range of interest for fluid viscous devices applied in the field of structural control).

Stability was analysed by applying harmonic actions, with frequency values ranging from 0.1 to 15 Hz, to the considered systems. The main outcome of this enquiry is represented by the reduction of the critical time step size $\Delta t^q$ relevant to the CD method, compared to the corresponding $\Delta t^x$ expression for linear viscous cases, given by [7]:

$$\Delta t^q = \frac{T_n}{\pi}$$

where $T_n$ is the smallest period of a $n$ degree-of-freedom system.

The following analytical relationship between $\Delta t^q$ and $\Delta t^x$ was derived from the previous enquiry:

$$\Delta t^q = \Delta t^x \left[ 1 - \left( \frac{q^q}{1.4 \Gamma(2-q)} - 0.5553 \right) \right] = \frac{T_n}{\pi} \left[ 1 - \left( \frac{q^q}{1.4 \Gamma(2-q)} - 0.5553 \right) \right]$$

(6)

A good correlation between the average values of the ratio $\Delta t^q / \Delta t^x$ numerically estimated from the analyses conducted on the three systems considered, and the $\Delta t^q / \Delta t^x$ normalized analytical curve derived from (6) is highlighted by the graph in Figure 2.
Concerning the IN method, unconditional stability properties were found also for the FrD-governed equations of motion.

The accuracy features of both algorithms resulted to be functions of the applied action, the mechanical parameters of the system, and the time step size. A demonstrative discussion about accuracy is reported in the following paragraph for a typical case study.

3 A case study

The application presented herein concerns a two-story frame including braces equipped with FV devices. This structure, schematized as a two degree-of-freedom system with the horizontal displacements $x_1$ and $x_2$ assumed as lagrangian coordinates, is characterized by the following story mass, global story stiffness, and damping coefficient values: $m_1 = m_2 = 37.7$ kg; $k_1 = k_2 = 68267$ N/mm, $c_1 = c_2 = 160$ Ns/mm; $c_1^* = c_2^* = 386$ Ns²/mm. The global story stiffness coefficients, $k_1$ and $k_2$, are obtained as the sum of the terms relevant to the frame ($k_1 = k_2 = 66596$ N/mm) and the elastic response components ($k_1^* = k_2^* = 1671$ N/mm) of the elastic-damping FV devices. The two vibration periods are: $T_1 = 0.25$ s and $T_2 = 0.09$ s.

Said $C$, $Z$, and $K$ the structural damping, supplementary damping, and global stiffness matrices:

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \quad Z = \begin{bmatrix} c_1^* + c_2^* & -c_2^* \\ -c_2^* & c_2^* \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{7}$$

and $F$ the dynamic action, the system equations of motion are:

$$\mathbf{M} \ddot{x} + \mathbf{C} \dot{x} + \mathbf{K} x + \mathbf{Z} \mathbf{D}^0 \left[ x \right] = F \tag{8}$$

where:

$$\mathbf{D}^0 \left[ x \right] = \begin{bmatrix} \frac{d^2 x_1}{dt^2} \\ \frac{d^2 x_2}{dt^2} \end{bmatrix}$$

represents the vector of the FrD functions of the displacement.

Since from the analyses discussed in the previous section the most critical conditions on stability of the CD scheme emerged for $q = 0.9$, this value was purposely selected within this case study.

Accuracy of the generalized formulations of the CD and IN methods was evaluated by comparing the maximum displacement values provided by the numerical computation with the analytical results separately obtained by means of a frequency domain transformation of equation (8):

$$X(\nu) = \mathbf{H}(\nu) \cdot F(\nu) \tag{9}$$
where: \( \nu = \) vibration frequency; \( X(\nu) = \) Fourier transform of the displacement response; \( F(\nu) = \) Fourier transform of the forcing action; \( H(\nu) = \) frequency response matrix of the system, given by:

\[
H(\nu) = \left[ -(2\pi \nu)^2 M + K + i(2\pi \nu)C + i^q(2\pi \nu)^{q} Z \right]^{-1}
\] (10)

being \( i^q \) the \( q \) power of the imaginary unit:

\[
i^q = \cos \left( \frac{\pi q}{2} \right) + i \sin \left( \frac{\pi q}{2} \right)
\]

The analytical expressions of the frequency response moduli (i.e., the moduli \( |H_{11}|, |H_{12}| = |H_{21}|, \) and \( |H_{22}| \) of the components of the \( H \) matrix) provide the exact response of the considered structure subjected to harmonic actions with amplitude equal to one. The values of \( |H_{11}|, |H_{12}|, \) and \( |H_{22}| \) were thereby compared with the corresponding \( |H_{n}^{num}| \) values obtained from the numerical integration developed by the CD and IN procedures. In order to satisfy the stability condition given by (6) for CD, time steps \( \Delta t = h \) no greater than 0.01 s were adopted.

As shown in Figure 3, where the results for the \( h = 0.01 \) s calculations are plotted for \( |H_{11}|, |H_{12}|, \) the correlation between numerical values and \( |H_{11}|, |H_{12}|, \) and \( |H_{22}| \) points is generally good. Nonetheless, by estimating the percent differences between the two sets of data:

\[
\Delta(\left|H_{\nu}\right|) = 100 \cdot \frac{|H_{\nu}^{num}| - |H_{\nu}|}{|H_{\nu}|}
\] (11)

a greater accuracy of the implicit procedure compared to the explicit one comes out, especially in the \( \nu > 5 \) Hz range. The approximation obviously improves when \( h \) is reduced from 0.01 s to 0.005 s (Figure 4). For the \( \nu \) range plotted in Figure 4 (where \( \Delta(\left|H_{22}\right|) \) is not reported since it practically coincides with \( \Delta(\left|H_{11}\right|) \) ), this last condition determines \( \Delta(\left|H_{\nu}\right|) \) values generally lower than 20\% (\( \Delta(\left|H_{12}\right|)_{\text{max}} = 17.3\%; \nu = 15 \) Hz) for the CD scheme, and no greater than 3.5\% (\( \Delta(\left|H_{11}\right|)_{\text{max}} \equiv \Delta(\left|H_{22}\right|)_{\text{max}} = 3.1\%; \nu = 7.5 \) Hz) for IN.

A measure of accuracy can also be obtained by evaluating the percent period elongation \( \Delta(\theta) \), i.e., the normalized difference between the fundamental vibration periods of the analytical and numerical solutions, \( T_{1} \) and \( T_{1}^{num} \):

\[
\Delta(\theta) = 100 \cdot \frac{T_{1}^{num} - T_{1}}{T_{1}}
\] (12)
The reduction of the step-size from 0.01 s to 0.005 s improves the approximation. In order to limit the $\Delta(\theta)$ ratio within a technically acceptable limit of 1% (i.e., the value suggested for linear problems [7], [8]) the time step should anyway be further restrained to $h = 0.001$ s. In fact, only in the $h \leq 0.001$ s range the harmful effect of reducing the number of points constituting each cycle of the harmonically oscillating action when the frequency increases, can be substantially controlled. This effect notably affects the accuracy of the numerical computation, and particularly for frequency values greater than 10 Hz.

4 Conclusive remarks

Special numerical methods are needed for solving the equations of motion of MDOF systems with damping properties expressed by means of fractional derivatives. Two procedures, representing modified versions of the explicit central difference and the implicit constant-average-acceleration methods, in which the LR-algorithm was included to perform FrD calculation, were proposed to this aim.

A critical analysis of the numerical properties of these two procedures was conducted which highlighted the following main aspects.

a) From the results of the analyses carried out on two, three, and four degrees-of-freedom systems, unconditional stability came out for the IN method also in the presence of FrD terms. Appreciable reductions of the critical step-size were instead found for the conditionally stable CD method compared to the linear viscous case, especially for $q$ values greater than 0.5.

b) Accuracy was assessed by estimating the percent differences between analytical and numerical values of the moduli of the frequency response matrix components, and the percent period elongation. By referring to the outcome of the presented case study, a remarkable influence of the input action frequency $\nu$ was observed. In particular, the numerical response obtained from both procedures resulted to be out of control for $\nu > 15$ Hz when $h \geq 0.001$ s values were adopted. Under this viewpoint, a considerable contribution to the loss of accuracy has to be reconduted to the harmful effect of reducing the number of points constituting each cycle of the harmonic action when the frequency increases.

c) Once satisfied the specified accuracy conditions (which, for the CD scheme, allow automatically satisfying even the stability conditions), the two integration methods show efficient algorithmic properties. Therefore, they can be regarded as usable tools for time domain analysis of systems with damping characteristics expressed by FrD relationships.

References


Figure 1: Schematic of $n$ degree-of-freedom dynamic system.

Figure 2: Analytical $\Delta t^{\text{cr}} / \Delta t_{\text{cr}}$ curve obtained from (6) and relevant numerical values.
Figure 3: Analytical curves (10) and numerical transfer function moduli of case study structure.

Figure 4: Percent differences between numerical and analytical transfer function moduli.