A study of the SP field produced by a polarized sphere in an electrically homogeneous and transversely anisotropic ground

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Abstract

The model of the polarized sphere, seated in an electrically homogeneous and isotropic half-space, has extensively been applied in the quantitative interpretation of self-potential (SP) anomalies, which are attributed to mineralized bodies, subsurface water or heat flow, or buried metallic bodies. Anisotropy, however, distorts the electric current flow and, consequently, the self-potential anomalies measured at ground surface. Important errors in the quantitative interpretation of the SP anomaly may be introduced, if electric anisotropy is not taken into account.

In the present paper, the analytical expression for the SP field of a polarized sphere is derived, considering an electrically homogeneous and transversely anisotropic ground. The behavior of the corresponding SP curve is studied and the errors in the quantitative interpretation are estimated. Finally, quantitative interpretation methods are proposed, to determine geometrical and polarization parameters of the polarized body.

The results and conclusions of this paper may be useful in environmental studies relevant to the detection of subsurface water/heat flow, fluid leakage and buried metallic bodies, or other hazardous environmental structures that can be simulated by a polarized sphere. They may also be useful in mineral exploration for graphite or sulfide ore bodies.
1. Introduction

The self-potential (SP) method has been extensively applied in the detection of mineralized bodies, subsurface water/heat flow, as well as buried metallic bodies. The quantitative interpretation of SP anomalies is usually carried out employing models of dipole current distribution in a homogeneous and isotropic medium. Less often, ground electrical inhomogeneity is taken into account, dividing the ground into isotropic parts with different resistivity.

The contribution of ground electrical anisotropy to the formation of SP anomalies has not been sufficiently studied, so far, and it is not taken into account in geophysical prospecting. Anisotropy may occur in sedimentary formations, where thin layers of different resistivity may alternate (macro-anisotropy), or in rocks which contain mineral grains, preferentially oriented with respect to their internal crystal structure (micro-anisotropy). In both cases, the flow of the electric current is distorted by the distribution of ground conductivity and significant errors in the quantitative interpretation of SP anomalies may result from not taking into account the ground anisotropy [1]. Therefore, ground anisotropy has to be introduced in modern methods of quantitative interpretation, which may help in making more reliable estimations about the parameters of the polarized body which produces the SP anomaly.

In the present paper, the SP field of a polarized sphere in an electrically homogeneous and transversely anisotropic ground is studied. Transverse anisotropy is the most usual type of anisotropy encountered in geoelectrical prospecting [2]. The expression for the SP anomaly at ground surface is derived and comments on the characteristics of the SP curve are made. Then, quantitative interpretation methods are proposed, in order to calculate the depth to the center of the sphere, the inclination of the polarization axis and the projection of the center at ground surface. The results and conclusions of the present paper may be useful in mineral and geothermal exploration, as well as in the detection of sources of environmentally hazardous activity, such as subsurface water leakage and buried metallic bodies.

2. The model of the polarized sphere in a homogeneous and transversely anisotropic ground.

The geometry of the problem is presented in fig.1. A polarized sphere of radius \( l \) is seated at depth \( h \) in a transversely anisotropic ground with anisotropy coefficient \( A \) and dip angle \( \theta \). The ground resistivity is \( \rho_\| \) at the plane of schistosity, or bedding, and \( \rho_\perp \) in the perpendicular direction. The polarization angle is \( \beta \). The vertical axis \( Oz \) separates the \( xOz \) plane in two half-planes, right and left. If the angle \( \beta \) belongs to the right half-plane, its value is positive. If it belongs to the left half-plane, its value is negative.

If the sphere is a metallic body, a vertical polarization tends to be developed, with a negative charge at the upper part of the body and a positive charge at the
lower part, because of the vertical gradient of the earth’s redox potential [3]. On the other hand, ionic currents tend to flow preferentially parallel to schistosity planes, because of the relatively high electric conductivity along these planes. The joint influence of these two factors, may, in the general case, result in a non vertical polarization, with an axis laid at the vertical plane xOz and a polarization angle $\theta$ with respect to vertical axis OZ.

If the polarization is developed by electrokinetic or thermoelectric coupling, between adjacent beds of different electrokinetic or thermoelectric coefficients, the polarization axis tends to be perpendicular to the plane of schistosity, or bedding. The preferential flow of ionic currents at the plane of schistosity, may displace the location of the poles, therefore the polarization axis is not generally, perpendicular to this plane.

In a homogeneous and isotropic ground, the exact expression for the electric potential of a polarized sphere is given as a series of Legendre polynomials. If, however, the radius $l$ is not more than one fifth of the depth $h$, a first order approximation may sufficiently describe the electric field [4]. Therefore, in geophysical prospecting, the self-potential $V(x)$ at ground surface is usually [5] given by

$$V(x) = M \frac{x \sin \theta + h \cos \theta}{(x^2 + h^2)^{3/2}}$$  \hspace{1cm} (1)

$M$ is the dipole moment of the sphere. Eqn (1), actually approximates the potential of a small polarized sphere with that of a dipole of length $2l$ ($l<<h$). An equivalent expression, which may also be usefull in the quantitative interpretation of SP anomalies, is given by [6]:

$$V(x) = M' \frac{x + h \cot \theta}{(x^2 + h^2)^{3/2}}$$  \hspace{1cm} (2)

where

$$M' = M \sin \theta$$  \hspace{1cm} (3)
Because the dipole approximation works satisfactorily in a homogeneous medium, it is reasonable to describe the SP field produced by a polarized sphere in a transversely anisotropic medium with a similar technique. Based on the following formula derived by Skianis & Hernández [1],

\[ V(x) = \frac{I \rho_m}{2\pi \sqrt{(\cos^2 a + A^2 \sin^2 a)x^2 + 2hx(1 - A^2) \sin a \cos a + (A^2 \cos^2 a + \sin^2 a)h^2}} \]  

(4)

\[ \rho_m = \sqrt{\rho_1 \rho_2} \]  

(5)

which is the expression for the potential produced by a point source \( I \) at depth \( h \) below the origin \( O \), in a transversely anisotropic ground, it can be proved (see Appendix 1), that the potential produced by a polarized sphere in such a ground is

\[ V(x) = M \frac{U \sin \theta + S(h \sin \theta + x \cos \theta) + Th \cos \theta}{(Ux^2 + 2Shx + Th^2)^{3/2}} \]  

(6)

where

\[ U = \cos^2 a + A^2 \sin^2 a \]  

(7)

\[ S = (1 - A^2) \sin a \cos a \]  

(8)

\[ T = \sin^2 a + A^2 \cos^2 a \]  

(9)

SP anomalies at a transversely anisotropic ground have to be interpreted according to eqn (6).

3. Comments on the behavior of the SP curve

The parameters \( A \) and \( a \) may have a significant influence on the conformation of the SP anomaly, which is expressed by eqn (6). If the polarization axis is vertical (\( \theta = 0^\circ \)), eqn (6) becomes

\[ V(x) = M \frac{Sx + Th}{(Ux^2 + 2Shx + Th^2)^{3/2}} \]  

(10)

In fig.2, it can be observed that a positive part appears, as if the polarization axis were inclined and seated in an isotropic medium. The parameters of the model of fig.2 are \( h = 20m, a = 30^\circ, A = 2, \theta = 0^\circ \). If the SP curve is interpreted in terms of a polarized sphere in an isotropic medium, according to De Witte’s method [6], the calculated values of \( h \) and \( \theta \) will be 23m and 36°, respectively. In other cases, the relative error at depth calculation may exceed the 100%.
If the vertically polarized sphere is seated in a ground with a vertical axis of anisotropy ($a = 0^\circ$), the expression for $V(x)$ may be derived by eqn (10), which gives

$$V(x) = M \frac{Ah}{(x^2 + A^2h^2)^{3/2}}$$  \hspace{1cm} (11)

which, compared with eqn (1), means that if the SP anomaly is interpreted with the assumption of an isotropic ground, the correct value $\theta = 0^\circ$ is expected to be found for the polarization angle, but the calculated depth will be $A$ times the true depth.

If the polarization axis of the sphere is not vertical and $a = 0^\circ$, both $h$ and $\theta$ will be erroneously calculated, if anisotropy is not taken into account.

The case of a polarization axis parallel to the schistosity ($\theta = 90^\circ - a$), may have a specific interest from the physical point of view. Based on eqn (6), it can be shown that the potential $V(x)$ is given by

$$V(x) = M \frac{x \sin\theta + h \cos\theta}{[A^2(x \cos\theta - h \sin\theta)^2 + (x \sin\theta + h \cos\theta)]^{3/2}}$$  \hspace{1cm} (12)

The positive part of the SP anomaly is not clearly expressed, as it can be seen in fig.3. On the contrary, it can be seen on the same figure, that the positive part is enhanced when $\theta$ lies on the left half-plane $xOz$.

It is obvious that anisotropy may significantly deform the SP anomaly measured at ground surface and introduce errors in the calculation of the
parameters of the polarized sphere. Therefore, modern interpretation methods have to be developed, taking into account the anisotropy of the ground.

4. Quantitative interpretation of the SP anomaly

The quantitative interpretation of the SP anomalies at an anisotropic ground aims at determining the depth \( h \), the polarization angle \( \theta \) and \( x_0 \) which is the projection of the centre of the sphere at ground surface. These parameters may be computed by an iterative trial and error algorithm, or directly, from the values of certain characteristic points of the SP anomaly.

4.1. Quantitative interpretation by a trial and error iterative procedure

The parameters \( h, \theta \) and \( x_0 \), as well as \( M, A \) and \( a \), may be found by an automated trial and error procedure, based on Marquardt’s algorithm [7]. The SP anomaly is interpreted according to

\[
V(x) = M \frac{U(x-x_0) \sin \theta + S[h \sin \theta + (x-x_0) \cos \theta] + Th \cos \theta}{\left[U(x-x_0)^2 + 2Sh(x-x_0) + Th^2\right]^{3/2}}
\]

which is a modification of eqn (6), taking into account that in field measurements the origin \( O \) of the axis is not necessarily the projection point of the center of the sphere at ground surface.

The input of the iterative algorithm are the initial values of \( h, \theta, x_0, M, A \) and \( a \), defined by the user. The iteration procedure may converge in a set of parameter values which correspond to a theoretical curve that fits satisfactorily the SP anomaly measured in the field. This set of values is the solution of the inverse problem.

Figure 4. Synthetic SP anomaly and its interpretation by Marquardt’s algorithm

The efficiency of this method was tested with a synthetic model, presented in fig.4. The quantitative interpretation was done with “Curve Expert” software package [8], which is based on Marquardt’s algorithm. It can be seen, from the table below, that there is a very good fit between the synthetic anomaly and the
curve which corresponds to the calculated parameters, but, although the initial guesses are not so far from the real values, the calculated parameters differ considerably from the parameters of the synthetic model.

Table 1. Efficiency test of the iterative method

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$h$</th>
<th>$\theta$</th>
<th>$x_0$</th>
<th>$A$</th>
<th>$a$</th>
<th>$M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>10.0m</td>
<td>50.0°</td>
<td>50.0m</td>
<td>2.00</td>
<td>30.0°</td>
<td>-10000mV.m²</td>
</tr>
<tr>
<td>Calculated</td>
<td>16.7m</td>
<td>21.3°</td>
<td>55.4m</td>
<td>1.35</td>
<td>77.5°</td>
<td>-12834mV.m²</td>
</tr>
<tr>
<td>Initial</td>
<td>15.7m</td>
<td>20.1°</td>
<td>60.0m</td>
<td>1.30</td>
<td>50.0°</td>
<td>-20000mV.m²</td>
</tr>
</tbody>
</table>

If the parameters $A$ and $a$ are known from geoelectrical measurements, both the negative and positive parts of the anomaly are clearly expressed and a better guess of the initial values of the parameters is done, the automated algorithm may converge in a more reliable solution. In practice, all these conditions are not often fulfilled. Therefore, Marquardt’s algorithm may provide a good fit between the SP anomaly and the theoretical curve, but not necessarily reliable results.

4.2. Direct interpretation of the SP anomaly

Equation (6) may be modified to (see Appendix 2)

$$V(x) = \frac{K}{[x + Sh / U + H \cot \omega]^{3/2}} \quad \text{(14)}$$

where

$$K = \frac{M(U \sin \theta + S \cos \theta)}{2U^{3/2}} \quad \text{(15)}$$

$$\cot \omega = \frac{A \cot \theta}{\cos^2 a + A^2 \sin^2 a + (1 - A^2) \sin a \cos a \cot \theta} \quad \text{(16)}$$

$$H = \frac{A}{\cos^2 a + A \sin^2 a} \quad \text{(17)}$$

It can be proved that the point $x_m$ of the maximum absolute value of the curve $V(x)$, which is expressed by eqn (14), is given by

$$x_m = H(-3 \cot \omega / 4 + \sqrt{9 \cot^2 \omega / 16 + 1/2}) + h(A^2 - 1) \sin a \cos a \cos^2 a + A \sin^2 a \quad \text{(18)}$$

The meaning of eqns (14), (16) and (17) is that the problem of the polarized sphere of polarization angle $\theta$ and depth $h$ in a transversely anisotropic ground is equivalent to the problem of the polarized sphere of depth $H$ and polarization angle $\omega$ in an isotropic ground. Based on this remark, the quantitative interpretation of the SP anomaly at a transversely anisotropic ground, if $A$ and $a$ are known, may be done according to the following procedure:
a) Calculate $H$ and $\omega$, using a direct quantitative interpretation method for a homogeneous ground (De Witte’s method [6] for example).

b) Calculate the depth $h$ and the polarization angle $\theta$, by the eqns

$$h = H \frac{\cos^2 \alpha + A^2 \sin^2 \alpha}{A}$$

and

$$\theta = \arctan \frac{A + \cot \omega \cdot (A^2 - 1) \sin \alpha \cdot \cos \alpha}{\cot \omega \cdot (\cos^2 \alpha + A^2 \sin^2 \alpha)}$$

(19) (20)

c) Find $x_0$ by

$$x_0 = x'_m - x_m$$

(21)

$x'_m$ is the maximum point for the origin of the SP profile measured in the field. $x_m$ is given by eqn (18) and it is the maximum point for an origin which coincides with the projection of the center of the sphere at ground surface.

This method was tested for the synthetic model of fig.4 and gave $h = 10m$, $\theta = 49^\circ$ and $x_0 = 51.3m$, which are very close to the true values ($h = 10m$, $\theta = 50^\circ$ and $x_0 = 50m$). The method worked so well because it was easy to define the maximum and minimum SP values of the synthetic model. In practice, the peaks of the SP anomaly can not always be detected precisely, therefore errors in the quantitative interpretation may be introduced.

5. Discussion and conclusions

Transverse anisotropy influences significantly the shape of the SP anomaly, therefore, it has to be taken into account in the computation of the parameters of the polarized sphere.

Automatic inversion, based on a trial and error iterative procedure, may give a solution which fits well the measured SP anomaly, but it may converge to parameter values which can be far away from reality.

Direct interpretation, based on characteristic points of the SP anomaly, may give more reliable results, if the negative and positive part of the SP curve are clearly expressed.

Transverse anisotropy may, in certain cases, suppress the negative or positive part of the SP anomaly. This does not favor a reliable quantitative interpretation. In cases that one of the peaks is not clearly expressed, a combination of a direct and an iterative inversion method could give more reliable results. In such a procedure, the input of the iterative algorithm can be the parameter values which are computed by the direct interpretation method.

The results and conclusions of this paper may be useful in environmental studies relevant to the detection of sources of subsurface heat flow, fluid leakage and buried metallic bodies. They may also be useful in mineral exploration for graphite or sulphide ore bodies.
6. References


7. Appendix 1. Derivation of eqn. (6)

The SP field of the polarized sphere of small radius \( l \) may be approximated by the field of a dipole \(-II\) with length \( 2l \) and pole coordinates \(-I (l \sin \theta, h - l \cos \theta)\) and \( I (l \sin \theta, h + l \cos \theta)\). Based on eqn. (4), and taking into account eqns (7), (8) and (9), the potential \( V(x) \) may be expressed as

\[
V(x) = \frac{-I \rho_m}{2 \pi \sqrt{2(2S(h-l \cos \theta)(x-l \sin \theta) + T(h-l \cos \theta))^2}} + \frac{I \rho_m}{2 \pi \sqrt{2(2S(h+l \cos \theta)(x+l \sin \theta) + T(h+l \cos \theta))^2}}
\]

(A1.1)

After some algebraic manipulation, where \( l \) is eliminated from the denominator because it is considered to be very small, it is deduced that

\[
V(x) \approx \frac{-I \rho_m}{2 \pi} \frac{U(x-l \sin \theta)^2 + 2S(h-l \cos \theta)(x-l \sin \theta) + T(h-l \cos \theta)^2}{2(Ux^2 + 2Shx + Th^2)^{3/2}} + \frac{I \rho_m}{2 \pi} \frac{U(x+l \sin \theta)^2 + 2S(h+l \cos \theta)(x+l \sin \theta) + T(h+l \cos \theta)^2}{2(Ux^2 + 2Shx + Th^2)^{3/2}}
\]

(A1.2)
Factors of the form $\sin^2 \theta$, $\cos^2 \theta$ and $\sin \theta \cos \theta$ may be eliminated from the numerator because of their very small magnitude and $V(x)$ may be expressed as

$$V(x) \approx \frac{-I_D m}{2 \pi} \frac{Ux \sin \theta + S(h \sin \theta + x \cos \theta) + Th \cos \theta}{(Ux^2 + 2Shx + Th^2)^{3/2}} \quad (A1.3)$$

The factor $[-I_D m/(2\pi)]2l$ is the dipole moment $M$, therefore

$$V(x) = M \frac{Ux \sin \theta + S(h \sin \theta + x \cos \theta) + Th \cos \theta}{(Ux^2 + 2Shx + Th^2)^{3/2}} \quad (A1.4)$$

**8. Appendix 2. Derivation of eqn (14)**

Equation (6) may take the form

$$V'(x) = M(U \sin \theta + S \cos \theta) \frac{x + \frac{Sh}{U} + h(\frac{S \sin \theta + T \cos \theta}{U \sin \theta + S \cos \theta} - \frac{S}{U})}{U^{3/2}} \quad (A2.1)$$

In order to derive this expression, the quantity $Sh/U$ was added and subtracted in the numerator of the fraction of eqn (6) and $S^2h^2/U^2$ was added and subtracted in the denominator of the same fraction.

Putting

$$K = \frac{M(U \sin \theta + S \cos \theta)}{U^{3/2}} \quad (A2.2)$$

$$H = h \sqrt{\frac{T}{U} - \frac{S^2}{U^2}} \quad (A2.3)$$

$$\cot \phi = \frac{S \sin \theta + T \cos \theta}{U \sin \theta + S \cos \theta} - \frac{S}{U} \quad (A2.4)$$

eqn (A2.1) may be transformed to

$$V(x) = K \frac{x + Sh/U + H \cot \phi}{[(x + Sh/U)^2 + H^2]^{3/2}} \quad (A2.5)$$

Recalling eqns (7), (8) and (9), eqns (A2.3) and (A2.4) may be expressed as

$$H = h \frac{A}{\cos^2 a + \sin^2 a} \quad (A2.6)$$

$$\cot \phi = \frac{A \cot \theta}{\cos^2 a + A^2 \sin^2 a + (1 - A^2) \sin a \cdot \cos a \cdot \cot \theta} \quad (A2.7)$$