Analysis of simple control policies for stormwater management in two connected dams at Mawson Lakes

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Abstract

We will analyse the management of stormwater storage in interconnected dams at Mawson Lakes. Mawson Lakes is a suburban housing development, situated 12 km North of the Adelaide CBD in South Australia. In this paper we have developed a discrete state model that consists of two connected dams. It is assumed that the input of stormwater into the first dam is stochastic with an arbitrary but known probability distribution and that there is regular release from the second dam that reflects the regular demand for water. We wish to find a control policy that releases an optimal flow of water from the first dam to the second dam to minimise the amount of overflow from the system. In the first instance we have restricted our attention to simple and practical classes of control policies. The cost of a particular policy will depend on the expected volume of water lost through overflow. An appropriate cost function will assist in determining an optimal pumping policy for our system. A numerical example is used to illustrate the theoretical solution presented in the paper. A simulation of our system of two connected dams enables us to illustrate the validity of our theoretical results.

1 Introduction

The work presented in this paper will assist us to understand the underlying behaviour of a simple water management system. By the year 2010 the Mawson Lakes suburb will provide for up to 3,500 dwellings and 10,000 residents. The intention is to capture and treat all the stormwater runoff from the surrounding catchment areas. The reclaimed water will be supplied to residential and commer-
cial sites for irrigation and other non-potable usage. The Mawson Lakes water cycle system will consist of a series of small dams, where the largest capacity will be 45 megalitres, determined from the expected daily demand for water. The daily demand will be approximately $3 - 4$ megalitres. Since this is a preliminary investigation we have been mainly concerned with the development of a discrete state model that considers a system of two connected dams. The discrete state model is practical for our system at Mawson Lakes, where the state of the system is the content level of the dams in megalitres and where the input is determined by the rainfall measurements in millimetres. In our model we will assume the input is directly proportional to rainfall. Over a short period of time, say one month, the daily rainfall can be approximated by independent and identically distributed random variables. We have not assumed any particular distribution except for the purpose of numerical illustration. However, studies by Rosenberg et al [5], suggest monthly rainfall totals can be modelled using gamma distributions and series of associated Laguerre polynomials.

1.1 Previous work

Previous studies by Moran [1, 2, 3] and Yeo [6, 7] have analysed a single storage system with independent and identically distributed inputs, occurring as a Poisson process. Simple rules were used to determine the instantaneous release rates and the expected average behaviour. These models provide a useful background for our analysis of connected storage systems.

2 Problem description

Consider a system of two small connected dams each of finite capacity. We denote the content of the first and second dams by $Z_1 \in \{0, 1, \ldots, h\}$ and $Z_2 \in \{0, 1, \ldots, k\}$ respectively. We assume a stochastic supply of water to the first dam and a regular demand for water from the second dam. The system is controlled by pumping water from the first to the second dam. We have formulated a discrete state model in which the state of the system, at time $t$, is an ordered pair $(Z_{1,t}, Z_{2,t})$, specifying the content of the two dams. We will consider classes of simple and practical control policies that depend only on the current state of the system. We wish to choose a control policy that releases an optimal flow of water from the first to the second dam, and minimises the expected volume of water lost through overflow.

2.1 Classes of control policies

In this paper we will describe four different classes of control policies and determine the amount of expected overflow from the system for each particular policy. This will enable us to compare the policies and choose the best practical management policy. For each control policy $\mathcal{P}$ there will be a unique probability associated with each state transition and a corresponding transition matrix, $H = H(\mathcal{P}) \in$
where \( n = (h + 1)(k + 1) \). We order the states \((0, 0), (1, 0), \ldots, (h, 0), (0, 1), (1, 1), \ldots, (h-1, k), (h, k)\), and number them from \( i = 0 \) to \((h+1)(k+1)\). The \((i, j)\)th element will be the probability of transition from state \( i \) to state \( j \). By examining the transition matrix, \( H \), we can determine the steady state behaviour of the system for a particular policy. The steady state \( x = x(P) \in \mathbb{R}^n \) is the vector of invariant state probabilities determined by the non-negative eigenvector of the transposed transition matrix \( K = H^T \) corresponding to the unit eigenvalue. Thus we find \( x \) by solving the equation

\[
K x = x \quad \text{subject to the conditions} \quad x \geq 0 \quad \text{and} \quad 1^T x = 1. \tag{1}
\]

The key finding is that the eigenvector of the large transposed matrix \( K \in \mathbb{R}^{n \times n} \) can be found from the corresponding eigenvector of a small block matrix of size \((h+1) \times (h+1)\). The matrix \( K \) can be written in block form as \( K = F + G \), where \( F, G \in \mathbb{R}^{n \times n} \) are block matrices with simple structure. For details we refer the reader to Piantadosi and Howlett [4]. By substituting \( y = (I-F)x \) and rearranging we obtain the matrix equation

\[
[I - G(I - F)^{-1}] y = 0. \tag{2}
\]

This effectively reduces our original problem to solving a matrix equation \([I - S]v = 0\) where \( S \) has a simple block structure and where \( v \) is a sub-vector of \( y \). Once again we refer to Piantadosi and Howlett [4] for details. The block matrix \( I - S \) is reduced to block upper triangular form using Gaussian elimination. It has been shown by Piantadosi and Howlett [4] that all blocks on the leading diagonal are non-singular except for the final block. Thus we have reduced our original problem to an eigenvalue problem for a much smaller matrix.

2.2 The transitions for each specific control policy

In this section we will describe the transitions for each class of control policies. In each case the conditional probability that these transitions will occur given that \( r \) units of water enter the system is denoted by \( p_r \). For policy class \( P_1 \) we select an integer \( m \in [1, h] \), where \( m \) is the fixed level of control for a particular policy, and assume regular demand of 1 unit. The transitions for this policy \( P_{1,m} \) are as follows:

- for the state \((z_1, 0)\) where \( z_1 < m \) we do not pump from either dam. If \( r \) units of water enter the first dam then \((z_1, 0) \rightarrow (\min([z_1 + r], h), 0)\). If \( z_1 + r > h \), then overflow occurs;
- for the state \((z_1, z_2)\) where \(z_1 < m\) and \(0 < z_2\) we do not pump water from the first dam but we do pump one unit from the second dam. If \(r\) units of water enter the first dam then \((z_1, z_2) \rightarrow (\min([z_1 + r], h), z_2 - 1)\). If \(z_1 + r > h\), then overflow occurs;
- for the state \((z_1, 0)\) where \(z_1 \geq m\) we pump \(m\) units from the first dam into the second dam. If \(r\) units of water enter the system then \((z_1, 0) \rightarrow (\min([z_1 - m + r], h), m)\). If \(z_1 - m + r > h\), then overflow occurs; and
- for the state \((z_1, z_2)\) where \(z_1 \geq m\) and \(0 < z_2\) we pump \(m\) units from the first dam into the second dam and pump one unit from the second dam to meet the regular demand. If \(r\) units of water enter the system then \((z_1, z_2) \rightarrow (\min([z_1 - m + r], h), \min(z_2 + m - 1, k))\). If \(z_1 - m + r > h\) or \(z_2 + m - 1 > k\), then overflow occurs.

The transitions for policy class \(P_2\) are the same as for policy class \(P_1\), except we do not allow the second dam to overflow. Therefore the last transition in policy \(P_{1,m}\) is replaced in policy \(P_{2,m}\) by the following two transitions:

- for the state \((z_1, z_2)\) where \(z_1 \geq m\), \(z_2 \leq k - m\) and \(0 < z_2\) we pump \(m\) units from the first dam into the second dam and pump one unit from the second dam to meet the regular demand. If \(r\) units of water enter the system then \((z_1, z_2) \rightarrow (\min([z_1 - m + r], h), z_2 + m - 1)\). If \(z_1 - m + r > h\), then overflow occurs; and
- for the state \((z_1, z_2)\) where \(z_1 \geq m\), \(z_2 > k - m\) and \(0 < z_2\) we do not pump any water from the first dam however we do pump one unit from the second dam. Thus if \(r\) units of water enter the system \((z_1, z_2) \rightarrow (\min([z_1 + r], h), z_2 - 1)\). If \(z_1 + r > h\), then overflow occurs.

The transitions for policy class \(P_3\) are the same as for policy class \(P_2\) however we try to keep the second dam as full as possible, therefore the final transition from policy \(P_{2,m}\) is replaced in \(P_{3,m}\) by:

- for the state \((z_1, z_2)\) where \(z_1 \geq m\), \(z_2 > k - m\) and \(0 < z_2\) we pump an amount \(m^* = k - z_2\) from the first dam and we pump one unit from the second dam. If \(r\) units of water enter the system then \((z_1, z_2) \rightarrow (\min([z_1 - m^* + r], h), \min([z_2 + m^* - 1], k))\). If \(z_1 - m^* + r > h\), then overflow occurs.

The probability transition matrices, \(H\) for each policy described above can be written in block matrix structure. This allows us to implement the methodology detailed in Section 2.1 to solve our system for a particular policy. We will also consider another class of policies \(P_4\) in Section 3, which has a slightly different block matrix structure. This class has only one member.

3 Numerical example

Consider a system of two dams with discrete states

\[ z_1 \in \{0, 1, 2, 3, 4, 5, 6\} \quad \text{and} \quad z_2 \in \{0, 1, 2, 3, 4\}. \]

Assume that the inflow to the first dam is defined by \(p_r = (0.5)^{r+1}\) for \(r = 0, 1, \ldots\) and consider each of the control policies described in Section 2.2. The transition
probability matrix for policy 1, where \( m = 2 \), has the block matrix form

\[
H = \begin{bmatrix}
A & 0 & B & 0 & 0 \\
A & 0 & B & 0 & 0 \\
0 & A & 0 & B & 0 \\
0 & 0 & A & 0 & B \\
0 & 0 & 0 & A & B \\
\end{bmatrix}
\]

\[
A = \begin{bmatrix}
p_0^T \\
p_1^T \\
0 \\
\vdots \\
0 \\
\end{bmatrix}_{7 \times 7}
\]

\[
B = \begin{bmatrix}
0 \\
p_0^T \\
p_1^T \\
\vdots \\
p_4^T \\
\end{bmatrix}_{7 \times 7}
\]

where \( p_0^T = [\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{64}] \), \( p_1^T = [0, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}] \), \( p_2^T = [0, 0, \frac{1}{2}, \ldots, \frac{1}{16}] \), etc.

As explained in Section 2.1 we solve an equation in the form

\[(I - S)v = 0\]

where \( S = [S_{ij}] \) for \( i, j = \{0, 1, 2\} \). Using the elimination process described by Piantadosi and Howlett [4] we find the reduced coefficient matrix

\[(I - S)v = 0 \rightarrow \begin{bmatrix} I & T_{01} & T_{02} \\
0 & I & T_{12} \\
0 & 0 & T_{22} \end{bmatrix} \begin{bmatrix} v_0 \\
v_1 \\
v_2 \end{bmatrix} = 0.
\]

Hence we solve the matrix equation

\[T_{22}v_2 = 0\]

to find an eigenvector \( v_2 \geq 0 \) and by back substitution

\[v_1 = -T_{12}v_2 \quad \text{and} \quad v_2 = -T_{01}v_1 - T_{02}v_2.
\]

In this case we obtain the steady state probability vector \( x \) given by

\[
\begin{bmatrix}
163 & 540 & 270 & 135 & 135 & 77 & 77 & 163 & 143 & 191 & 167 & 626 \\
5918 & 6853 & 6853 & 6853 & 13766 & 15635 & 15635 & 5918 & 2790 & 7453 & 13033 & 15683 \\
2066 & 52132 & 52132 & 2959 & 2201 & 377 & 8076 & 7429 & 7429 & 5161 & 2915 \\
134 & 90 & 45 & 45 & 176 & 156 & 371 & 14717 & 22691 & 22691 & 22691 \\
5507 & 5441 & 5441 & 5441 & 3939 & 7503 & 337 & 337 & 22691 & 22691 & 22691 \end{bmatrix}^T.
\]

Using the steady state vector \( x \) we can calculate the expected overflow of water from the system for a particular policy. Let \( z = z(s) = (z_1(s), z_2(s)) \) for \( s = 1, 2, \ldots, n \) denote the collection of all possible states. The expected overflow is calculated by

\[
J = \sum_{s=1}^{n} \left[ \sum_{r=0}^{\infty} f[z(s)|r]p_r \right] x_s
\]

where \( f[z(s)|r] \) is the overflow from state \( z(s) \) when \( r \) units of water enter the first dam. We will consider the same pumping policy for four different values \( m = \ldots, \).
1, 2, 3, 4 of the control parameter. We obtain the steady state vector \( x = x[m] \) for each particular value of the control parameter and determine the expected total overflow in each case.

Table 1: Expected overflow from the system for policy \( P_1 \).

<table>
<thead>
<tr>
<th></th>
<th>( m = 1 )</th>
<th>( m = 2 )</th>
<th>( m = 3 )</th>
<th>( m = 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_1 )</td>
<td>0.1429</td>
<td>0.0388</td>
<td>0.0400</td>
<td>0.0588</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>0</td>
<td>0.1451</td>
<td>0.1948</td>
<td>0.2096</td>
</tr>
<tr>
<td>Total</td>
<td>0.1429</td>
<td>0.1839</td>
<td>0.2348</td>
<td>0.2684</td>
</tr>
</tbody>
</table>

Table 1 compares the parameter values for the first class of control policies by considering the overflow \( J_i = J_i[m] \) for each \( i = 1, 2 \) from the first and second dams. We have calculated the expected overflow from the system for the second and third class of control policies and compared these results to those from the first class. We note that the case \( m = 1 \) is the same for each class. The results show that when comparing the case \( m = 2 \) for each of the classes the third class produces the least amount of overflow from the system. The results suggest that the best policy may be to keep the second dam full.

Now let us consider a new control policy where we pump water from the first dam provided the first dam is not empty without causing the second dam to overflow. The same methodology described in Section 2.1 can be used to calculate the steady state probability vector. By taking the same numerical example we can compare the results to the previous policies.

Table 2: Expected overflow from the system for each class of control policies.

<table>
<thead>
<tr>
<th></th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 2 )</td>
<td>( m = 2 )</td>
<td>( m = 2 )</td>
<td>( m = 2 )</td>
<td>( m = 2 )</td>
</tr>
<tr>
<td>( J_1 )</td>
<td>0.0388</td>
<td>0.1188</td>
<td>0.1208</td>
<td>0.1014</td>
</tr>
<tr>
<td>( J_2 )</td>
<td>0.1451</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>0.1839</td>
<td>0.1188</td>
<td>0.1208</td>
<td>0.1014</td>
</tr>
</tbody>
</table>

Table 2 compares best case \( m = 2 \) which produces the least expected overflow for the first, second and third policies with the expected overflow from the fourth policy. It is evident that policy \( P_4 \), to keep the second dam full, results in the least amount of overflow from the system. In our model we have ignored pumping costs.
In a real system there are likely to be other cost factors to consider. It is possible that less frequent pumping of larger volumes may be more economical.

4 A simulation of the two connected dam system

We have developed a Scalable Vector Graphics (SVG) simulation. It simulates the our system of two connected dams for each of the described policies. Simulated pseudo-random rainfall fills the first dam, water is pumped from the first to the second using a simple control policy and water is drawn from the second dam at a constant rate. The simulation shows the level of water in each dam and shows the water being pumped from the first and second dams. We have a set of histograms that represents the relative frequencies of occurrence for the different states of the system \((z_1, z_2)\). In the long run we approximate the steady state probability vector for the system, for a particular policy. We have simulated the example described in Section 3. Figure 1 displays the simulation for this particular example.

In the first instance we ran the simulation and obtained the steady state probability vector for the first policy when \(m = 2\). This allows us to compare the theoretical calculated steady state vector with the simulated steady state probability vector. Now we can run the simulation for the second and third classes of control policies and compare the results.

5 Summary

We have implemented a general method of analysis for classes of simple control policies in a system of two connected dams where we assume a stochastic supply
Figure 2: Compares the theoretical versus the simulated steady state probabilities for the first class with $m = 2$.

Figure 3: Compares the steady state probabilities of the theoretical versus the simulated for the second class of control policies with $m = 2$.

and regular demand. We calculated steady state probabilities for each particular policy and hence determined the expected overflow from the system. A key finding is that calculation of the steady state probability vector for a large system can be reduced to a much smaller calculation using the block matrix structure. Ultimately we would like to extend our model to include more complicated connections and the delays associated with the treatment of stormwater.

We have developed a simulation of the system of two connected dams and found that the simulated results are comparable to the theoretical results. We wish to develop a simulation of a realistic model of the system at Mawson Lakes.

We also believe a similar analysis is possible for the policies considered in this paper when a continuous state space is used for the first dam. The matrices must be replaced by linear integral operators but the overall block structure remains the same.
Figure 4: Compares the steady state probability vector of the theoretical versus the simulated for the third class of control policies with $m = 2$.

Figure 5: Compares the steady state probability vector of the theoretical versus the simulated for the fourth control policy.

References


