Moving boundary problems for parabolic equations

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Abstract

Parabolic problems are considered in a noncylindrical domain with non-smooth with respect to time “lateral boundary”. The latter consists of two nonintersecting parts: Dirichlet data are given on the first one and oblique derivation data are given on the second one. By using single-layer potentials, these problems are solved in anisotropic Hölder spaces.

1 Introduction

Let $0 < \alpha < 1$. Assuming that $\Omega$ in an open set of the space $\mathbb{R}^{n+1}$ ($n \geq 1$) of variables $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, we let $C^{0,\alpha}(\overline{\Omega})$ denote the anisotropic Hölder space consisting of those functions $u$ that satisfy

$$\|u; \Omega\|^{0,\alpha} := \sup_{(x,t) \in \Omega} |u(x,t)| + \sup_{(x,t),(y,s) \in \Omega, (x,t) \neq (y,s)} \frac{|u(x,t) - u(y,s)|}{|x-y|^{\alpha} + |t-s|^{\alpha/2}} < +\infty.$$
Next, we let $C^{1,\alpha}(\overline{\Omega})$ denote the anisotropic Hölder space of those functions $u$ that satisfy

$$
\|u; \Omega\|^{1,\alpha} := \sup_{(x,t) \in \Omega} |u(x, t)| + 
\sup_{(x,t),(x,s) \in \Omega, t \neq s} \frac{|u(x, t) - u(x, s)|}{|t - s|^{1+\alpha}/2} + \sum_{i=1}^{n} \|\partial_i u; \Omega\|^{0,\alpha}
< +\infty,
$$

where $\partial_i u := \frac{\partial u}{\partial x_i}$.

We consider a domain $\Omega \subset \mathbb{R}^{n+1}_T := \mathbb{R}^{n} \times ]0,T[, 0 < T < +\infty$, with boundary $\partial \Omega = \partial_0 \Omega \cup \partial_T \Omega \cup \Sigma$, where $\partial_0 \Omega \subset \mathbb{R}^{n}$ is a domain in the plane $\{t = 0\}$, $\partial_T \Omega \subset \mathbb{R}^{n}$ is a domain in the plane $\{t = T\}$, and $\Sigma$ is $n$-dimensional surface ("lateral boundary" of $\Omega$). We suppose that $\Sigma$ is compact and that

$$
\Sigma = \Sigma_0 \cup \Sigma_1, \text{ where } \Sigma_0 \cap \Sigma_1 = \emptyset \quad \text{(Fig. 1)}.
$$

Assume that at each point $P^0 = (x^0, t^0) \in \Sigma$ the intersection

$$
\Sigma(t^0) := \Sigma \cap \{t = t^0\}
$$

has the unit inner normal

$$
n(P^0) = (\overline{n}(P^0), 0), \text{ where } \overline{n}(P^0) = (n_1(P^0), \ldots, n_n(P^0)),
$$

lying in the plane $\{t = t^0\}$. Let $\{e_1(P^0), \ldots, e_n(P^0), e_{n+1}\}$ denote an orthonormal basis in $\mathbb{R}^{n+1}$ with the origin $(x^0, 0)$, where $e_n(P^0) =$
Figure 1: Noncylindrical domain $\Omega$ with non-smooth "lateral boundary" $\Sigma = \Sigma_0 \cup \Sigma_1$
We say that the surface \( \Sigma \) belongs to the anisotropic Hölder class \( C^{1,\alpha} \) if there exists a number \( d > 0 \) such that each point \( P \in \Sigma \) has a neighborhood \( O(P) \) with the property: the intersection \( \Sigma \cap O(P) \) can be written in the \( P \)-coordinate system as

\[
\Sigma \cap O(P) = \{(y', y_n, t) \in \mathbb{R}^{n+1}_T; (y', t) \in \overline{B}(P, d), y_n = g((y', t); P)\},
\]

where the function \( g(\cdot, P) \in C^{1,\alpha}(\overline{B}(P, d)) \).

Suppose that \( S \in C^k \). For each \( P \in \Omega \), let the mapping \( f(\cdot; P) : \overline{B}(P, d) \to \Sigma \cap O(P) \) be defined by \( f((y', t), P) := (y', g(y', t; P), t) \) for all \( (y', t) \in \overline{B}(P, d) \). We say that a function \( \varphi : \Sigma \to \mathbb{R} \) belongs to the anisotropic Hölder space \( C^{k,\alpha}(\Sigma) \), \( k = 0, 1 \), if

\[
\|\varphi; \Sigma\|^k,\alpha := \sup_{P \in \Sigma} \|\varphi \circ f(\cdot; P); B(P, d)\|^k,\alpha < +\infty.
\]

Furthermore, we denote

\[
C^{k,\alpha}(\Sigma) := \{\varphi \in C^{k,\alpha}(\Sigma); \varphi(x, 0) = 0\}.
\]

In the domain \( \Omega \) we consider the mixed boundary value problem:

\[
\begin{cases}
\partial_t u - \Delta u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega_0, \\
u &= \psi_0 \quad \text{on } \Sigma_0, \quad b \cdot \nabla u = \psi_1 \quad \text{on } \Sigma_1,
\end{cases}
\]

(1)

where \( b = (b_1, \ldots, b_n) \), \( \partial = (\partial_1, \ldots, \partial_n) \); assuming that

\[
b \cdot \overline{n} \neq 0 \text{ on } \Sigma_1 \text{ and } b_i \in C^{\alpha}(\Sigma).
\]

(2)

We prove in Section 2 that the sum of two single-layer potentials corresponding to the surfaces \( \Sigma_0 \) and \( \Sigma_1 \) respectively, is the regular solution of problem (1), provided that the couple of densities of these potentials is the solution of the corresponding system of two boundary integral equations.

In the same way, we solve in Section 3 the mixed problem for a linear parabolic equation with variable coefficients.

Remarks

(1) Obviously, \( \Sigma \in C^{1,\alpha} \) if and only if \( \Sigma_k \in C^{1,\alpha}(k = 0, 1) \).

(2) The domain \( \Omega \) may be unbounded.

(3) If \( \Sigma = \Sigma_0 \) (the Dirichlet problem) or \( \Sigma = \Sigma_1 \) (the oblique derivative problem), the result follows from Baderko[1].
2 The mixed problem for the heat equation

To solve problem (1), we use two single-layer potentials

\[(U_k \varphi)(x, t) := \int_0^t d\tau \int_{\Sigma_k(\tau)} Z(x - \xi, t - \tau) \varphi_k(\xi, \tau) d\xi\]

for \((x, t) \in \mathbb{R}^{n+1}, k = 0, 1\), where the function

\[Z(x, t) := \frac{1}{2\sqrt{\pi t}} \exp \left\{ -\frac{x^2}{4t} \right\}, \quad x \in \mathbb{R}^n, \quad t > 0,\]

is the Green function of the Cauchy problem for heat equation.

The regularity properties of these potentials, see Baderko[1], imply that, for all \(\varphi_k \in C^\alpha(\Sigma_k), k = 0, 1\) the sum \(U_k \varphi_k\) (the repeated index convention for summation is systematically used) is a regular solution of the heat equation in \(\Omega\), and \(U_k \varphi_k\) belongs to \(C^{1,\alpha}(\bar{\Omega})\) with the estimate:

\[\|U_k \varphi_k; \Omega\|^{1,\alpha} \leq \text{const} \|\varphi_k; \Sigma_k\|^{0,\alpha};\]

furthermore

\[b(P^0) \cdot (\partial U_k \varphi_k)(P^0) := \lim_{P \to P^0} b(P) \cdot (\partial U_k \varphi_k)(P)\]

\[= \frac{b(P^0) \cdot \overline{n}(P^0)}{2} \varphi_1(P^0) + \int_0^t d\tau \int_{\Sigma_k(\tau)} b(P^0) \cdot \partial_x Z(x^0 - \xi, t^0 - \tau) \varphi_k(\xi, \tau) d\xi\]

for all \(P^0 = (x^0, t^0) \in \Sigma_1\).

**Lemma 1** Assume that \(\Sigma \in C^{1,\alpha}\) and condition (2) is satisfied. Then, for all \(\psi_0 \in C^{1,\alpha}(\Sigma_0), \psi_1 \in C^{0,\alpha}(\Sigma_1)\), the system of boundary integral equations:

\[U_k \varphi_k = \psi_0 \text{ on } \Sigma_0, \quad b \cdot \partial U_k \varphi_k = \psi_1 \text{ on } \Sigma_1,\]

has an unique solution \(\{\varphi_k \in C^{0,\alpha}(\Sigma_k), k = 0, 1\}\); and the estimate holds:

\[\|\varphi_k; \Sigma_k\|^{0,\alpha} \leq \text{const} \{\|\psi_0; \Sigma_0\|^{1,\alpha} + \|\psi_1; \Sigma_1\|^{0,\alpha}\}.\]
Proof. The system

\[ U_0 \varphi_0 = \overline{\psi}_0 \text{ on } \Sigma_0, \quad b \cdot \partial U_1 \varphi_1 = \overline{\psi}_1 \text{ on } \Sigma_1, \]

for all \( \varphi_0 \in C^{1,\alpha}(\Sigma_0) \) and \( \varphi_1 \in C^{0,\alpha}(\Sigma_1) \), has the unique solution \( \{ \overline{\varphi}_k \in C^{0,\alpha}(\Sigma_k), k = 0, 1 \} \), with the corresponding estimate, see Baderko [1]. Since the kernels of both "additionnal" integrals (\( U_1 \varphi_1 \) on \( \Sigma_0 \)) and (\( b \cdot \partial U_0 \varphi_0 \) on \( \Sigma_1 \)) are continuous, the assertion follows.

**Theorem 2** Assume that the conditions of Lemma 1 are satisfied. Then:

1. problem (1) has the unique regular solution \( u \in C^{1,\alpha}(\overline{\Omega}) \);
2. this solution is of the form \( u = U_k \varphi_k \), where \( \{ \varphi_k \in C^{0,\alpha}(\Sigma_k), k = 0, 1 \} \) is the unique solution of boundary integral system (3);
3. the estimate holds

\[ \|u; \Omega\|^{1,\alpha} \leq \text{const} \left\{ \|\psi_0; \Sigma_0\|^{1,\alpha} + \|\psi_1; \Sigma_1\|^{0,\alpha} \right\}. \]

The assertion of theorem follows from Lemma 1 and regularity properties of potentials \( U_k \varphi_k \).

**3 The mixed problem for the parabolic equation**

In the domain \( \Omega \) we consider now the mixed problem for parabolic equation:

\[
\begin{cases}
\partial_t u - a_{ij} \partial_{ij} u + a_i \partial_i u + a_0 u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \Omega_0,
\end{cases}
\]

\[ u = \psi_0 \quad \text{on } \Sigma_0, \quad b \cdot \partial u = \psi_1 \quad \text{on } \Sigma_1, \]

assuming that

\[
\begin{cases}
\exists \delta > 0, \quad a_{ij}(P) \sigma_i \sigma_j \geq \delta \sigma^2 \text{ for all } P \in \mathbb{R}^{n+1}_T, \sigma \in \mathbb{R}^n, \\
a_{ij}, a_i, a_0 \in C^{0,\alpha}(\mathbb{R}^{n+1}_T).
\end{cases}
\]

To solve problem (4), we use the two appropriate single-layer potentials

\[
(U_k^* \varphi_k)(x, t) := \int_0^t dt \int_{\Sigma_k(\tau)} \Gamma(x, t; \xi, \tau) \varphi_k(\xi, \tau) ds_\xi
\]
for \((x,t) \in \mathbb{R}^{n+1}, k = 0, 1\), where \(\Gamma\) is the Green function of the Cauchy problem for the parabolic equation, see Friedman [2].

**Theorem 3** Assume that \(\Sigma \in C^{1,\alpha}\) and conditions (2),(4) are satisfied. Then:

1. for all \(\psi_0 \in C^{1,\alpha}(\Sigma_0), \psi_1 \in C^0,\alpha(\Sigma_1)\), problem (4) has the unique regular solution \(u \in C^{1,\alpha}(\bar{\Omega})\);

2. this solution is of the form: \(u = U_k^* \varphi_k\), where \(\{\varphi_k \in C^0,\alpha(\Sigma_k), k = 0, 1\}\) is the unique solution of the boundary integral system:

\[
U_k^* \varphi_k = \psi_0 \text{ on } \Sigma_0, \quad b \cdot \partial U_k^* \varphi_k = \psi_1 \text{ on } \Sigma_1;
\]

3. the estimate holds:

\[
\|u; \Omega\|^{1,\alpha} \leq \text{const}\{\|\psi_0; \Sigma_0\|^{1,\alpha} + \|\psi_1; \Sigma_1\|^{0,\alpha}\}.
\]

The proof of Theorem 3 is similar to that of Theorem 2.

**References**
