Scattering of waves by buried cavities via boundary element method

A.J.B. Tadeu & F.A.S. Paulo

Department of Civil Engineering, University of Coimbra, 3020 Coimbra, Portugal
Email: construc@cygnus.ci.uc.pt

Abstract

The Boundary Element Method (BEM) is used to determine the scattering of elastic waves by cavities buried in a two-dimensional elastic medium when struck by dynamic line sources. The BEM is formulated in the frequency domain by means of Green’s functions. The boundary of the cavity is discretized using constant, linear and quadratic elements.

The element in the full system of equations generated by BEM results from integrations of the Green’s functions and its derivatives along the discretized boundaries. The diagonal terms are the dominant ones in this system of equations, and therefore its solution is highly dependent on those terms. The integrations needed to produce the diagonal terms involve singularities. Thus, the integration of those singularities plays a key role in determining the accuracy of the final solution. This paper presents the evaluation of the singular integrals in analytical form. The BEM results are then compared with the analytical solutions of the cylindrical cavities.

Introduction

Many studies have been done on wave propagation in or around geologic and topographic irregularities in soil media including horizontally stratified deposits. Early studies on wave diffraction and scattering were concerned with wave motion and reverberations in regular alluvial basins (e.g. Wong\textsuperscript{16}) and with cavity induced wave scattering (e.g. Lee\textsuperscript{8}). Semi-analytical methods have recently been used to study wave diffraction caused by geological irregularities of arbitrary shape within globally homogeneous media (e.g. Lee\textsuperscript{9}). By contrast, the application of purely
numerical methods (i.e. finite elements or differences combined with transmitting boundaries) has mostly been restricted to situations where the response is required only within localized irregular domains; e.g. soils structure interaction problems (Kausel*). Discrete methods have also been used to model large alluvial basins, but only in plane-strain. Finally, hybrid methods involving a combination of finite elements to model the interior domain containing the inhomogeneities and semi-analytical representations for the exterior domain have been used (e.g. Sanchez^13).

Possibly the best means of analyzing wave propagation problems in unbounded infinitely large media is the boundary element method (BEM) because it satisfies the far field radiation conditions and allows description of the medium only at material discontinuities.

The BEM needs appropriate fundamental solutions, or Green's functions, relating the field variables (stresses, displacements) in a homogeneous medium to sources placed somewhere in the medium. The fundamental solution most often used is that of an infinite homogeneous space, because it is known in closed-form and has a relatively simple structure. BEM based on the Green's functions for either an infinite space or a half-space has been used to solve several problems involving diffraction of waves by surface irregularities of arbitrary shape. (e.g. Dravinski^4, Pedersen^12), cavities and buried structures (e.g. Beskos^2).

A major difficulty associated with the application of the BEM lies in the evaluation of the singular integrals (e.g. Hall^3, Kawase^7, Tadeu^14,15). Below, the direct BEM is used together with the two-dimensional full-space Green's functions to evaluate the displacement field associated with waves illuminating a cavity within a homogeneous elastic medium (i.e. a full-space). The problem is formulated in the frequency domain, and it presents the derivation of analytical expressions for the singular integrals. The boundary elements used are constant, linear and quadratic.

### Boundary Element Formulation

The scattering of waves by buried cavities via BEM requires evaluation of the integral (e.g. Manolis^10)

\[
H_{ij}^{kl} = \int_{C_l} H_{ij}(x_k, x_l, n_i)dC_l \tag{1}
\]

in which \(H_{ij}(x_k, x_l, n_i)\) is the component of the Green's tensor for traction components at \(x_k\) in direction \(i\) due to a concentrated load at \(x_l\) in direction \(j\), and \(n_i\) is the unit outward normal for the \(l^{th}\) boundary segment \(C_l\).
In the case of a medium under plane-strain conditions subjected to anti-plane and in-plane line loads, the required Green's functions are as follows (e.g. Dominguez):

\[ G_{33}(x, x_0) = i/(4\mu)H_0(k_\beta r) \]

\[ G_{ij}(x, x_0) = i/(4\mu)\left\{-\delta_{ij}H_0(k_\beta r) + \delta_{ij}/(k_\beta r)\left[H_1(k_\beta r) - \gamma H_1(k_\alpha r)\right]\right\} \]

\[ -i/(4\mu)\left[\partial r/\partial x_i \partial r/\partial x_j \left[H_2(k_\beta r) - \gamma^2 H_2(k_\alpha r)\right]\right] (i, j = 1, 2) \]

In these equations \( \gamma = \beta/\alpha \), where \( \alpha = \sqrt{(\lambda + 2\mu)/\rho} \) and \( \beta = \sqrt{\mu/\rho} \) are the velocities of compressional and shear waves, respectively; \( \lambda \) and \( \mu \) are the Lamé constants; \( \rho \) is the mass density; \( k_\alpha = \omega/\alpha \) and \( k_\beta = \omega/\beta \) are the wave-numbers; \( \omega \) is the circular frequency; \( r = |x-x_0| \) is the source-receiver distance; \( \delta_{ij} \) Kronecker's delta; the \( i = V-T; H(\cdot) \) are Hankel functions of the second kind and \( n^{th} \) order.

The expressions for the tractions \( H_{ij} \), may be obtained from \( G_{ij} \) by taking partial derivatives to deduce the strains and then applying Hooke's law to obtain the stresses. They are omitted here for the sake of brevity.

**Element Integration**

When the element to be integrated in eqn (1) is not the loaded element, the integrands are non-singular and the integrations are carried out using standard Gaussian quadrature. For the loaded element however, the integrands exhibit a singularity, but it is then possible to carry out the integrations in closed form. To demonstrate this assertion, consider the element of length \( L \) shown in Figure 1.

![Local coordinate system for element integration.](image)

**Figure 1**: Local coordinate system for element integration.
Since for this case \( r.nf=0 \), the singular terms \( H_{11}^{11} \) vanish. Consider next the integration of \( H_{ij}^{11} \) for \( i,j=1,2 \), which may be written as a combination of integrations of strains \( (\partial G_{11}/\partial x, \partial G_{11}/\partial y, \partial G_{22}/\partial y, \partial G_{12}/\partial x, \partial G_{12}/\partial y) \).

Let us perform the integration of the \( \partial G_{11}/\partial x \) along the constant, linear and quadratic elements. We may verify that \( \partial G_{11}/\partial x \) takes the form

\[
\frac{\partial G_{11}}{\partial x} = \frac{i}{4\mu} \left[ \left( C1 + C2 + C3 - \left( \frac{\partial r}{\partial x} \right)^2 C4 + 2 \left( \frac{\partial r}{\partial x} \right)^2 C5 \right) \frac{\partial r}{\partial x} - 2 \frac{\partial r}{\partial x} C5 \right]
\]

where

\[
C1 = k_p H_0(k_p r)
\]

\[
C2 = -1/(k_p r^2) \left[ H_1(k_p r) - \gamma H_1(k_r) \right]
\]

\[
C3 = 1/k_p r \left[ -H_1(k_p r)/r + k_p H_0(k_p r) - \gamma (-H_1(k_r)/r + k_r H_0(k_r)) \right]
\]

\[
C4 = -2/r \left[ H_2(k_p r) + k_p H_1(k_p r) - \gamma^2 \left( -2/r H_2(k_r) + k_r H_1(k_r) \right) \right]
\]

\[
C5 = \left[ H_2(k_p r)/r - \gamma^2 r^2 H_2(k_r)/r \right]
\]

The partial derivative \( \partial r/\partial x \), in eqn (4) represents the slope of the line connecting the nodal point with the integration point. Since for the loaded element it equals the slope of the segment, it follows that it is constant and does not affect the integration.

Given the recurrence relations, the ascending series, the limiting forms for Bessel functions, and the expressions (e.g. Abramowitz),

\[
\int_0^z g_0(t) dt = z g_0(z) + \pi z/2 \left( S_0(z) g_1(z) - S_1(z) g_0(z) \right)
\]

\[
\int_0^z t^\nu J_{\nu+1}(t) dt = \frac{1}{2^\nu \Gamma(\nu+1)} - z^\nu J_{\nu}(z)
\]

\[
\int_0^z t J_0(t) dt = z J_1(z)
\]

\[
\int_0^z t Y_0(t) dt = z Y_1(z) + 2/\pi
\]

where \( g_0(\ ) \) and \( g_1(\ ) \) are Struve functions, \( J_\nu(\ ) \) and \( Y_\nu(\ ) \) are \( \nu \)th order Bessel functions of the first and second kind respectively, \( S_0(\ ) \) denotes \( J_0(\ ) \) or \( Y_0(\ ) \), and \( \Gamma(\ ) \) is the Gamma function.

After mathematical manipulations (omitted here) one may obtain,
\[ \int_0^L \frac{\partial G}{\partial x} \, dr = \frac{i}{4\mu} \left[ -S1 + S2 - \left( \frac{\partial r}{\partial x} \right)^2 S3 + \frac{1}{2}(1 + \gamma^2) + 2 \left( \frac{\partial r}{\partial x} \right)^2 - 1 \right] S4 \left( \frac{\partial r}{\partial x} \right) \cdot \frac{\partial r}{\partial x} \right] \]  

(9)

where \( B_1 = -C12 + (\partial r/\partial x)^2 C13 \)

with \( C12 = C11(b) - \gamma C11(a) - \frac{1}{k_p} [H_1(b) - \gamma H_1(a)] \)

\( C13 = C11(b) - \gamma C11(a) - \frac{2}{k_p} [H_1(b) - \gamma H_1(a)] \)

\[ \int_0^L \frac{\partial G}{\partial x} \, dr = \frac{i}{4\mu} \left[ L \left( -S1 + S2 - \left( \frac{\partial r}{\partial x} \right)^2 S3 \right) - B_1 + 2 \left( \frac{\partial r}{\partial x} \right)^2 - 1 \right] C13 \left( \frac{\partial r}{\partial x} \right) \cdot \frac{\partial r}{\partial x} \right] \]  

(10)

The integrations of the other strains (\( \partial G_{11}/\partial y \), \( \partial G_{22}/\partial y \), \( \partial G_{21}/\partial x \), \( \partial G_{12}/\partial x \), \( \partial G_{22}/\partial y \)) can be obtained by similar manipulations. Equations (9), (10) and (11) provide the exact integrals for the singular terms associated with line sources in a two-dimensional homogeneous space (plane-strain).
Application of the Algorithm

The method and expressions described previously were implemented and validated by applying them to a cylindrical cavity in an infinite, homogeneous space subjected to SH and SV-P waves as shown in Figure 2.

The motion components are then computed and recorded at the receivers, both with the implemented constant, linear and quadratic boundary elements and with the analytical solution reported by Pao & Mow. The results are computed at 32 frequencies in the range from 2 to 64 Hz, that is, for shear (SV, SH) waves whose wavelengths are, respectively, between 10.0 and 0.3125 times the radius of the cylinder. The cavity is first modelled with 5 boundary elements. The number of boundary elements is then increased until it reaches a total of 50 elements. Each element represents a part of the surface of the cylinder whose length ranges from 6.28 to 0.628m.

The cavity is first subject to SH waves caused by an SH (anti-plane) line source at point O. Figure 3 displays the modulus of the amplitude of the response recorded at the receiver 2 produced by harmonically vibrating sources for two values of the excitation frequency, namely, 2 and 60 Hz, which correspond to incident waves with wavelengths of 50 and 1.66m respectively. Comparison of the results shows that the response improves as the number of the nodes in the boundary elements and the number of elements used to model the cavity increase. This behaviour was expected because the accuracy of the BEM solution depends on the ratio between the length of the incident waves and the length of the boundary elements. In our example, this ratio increases from 7.96 to 79.62 for a frequency of 2 Hz and from 0.27 to 2.65 for a
frequency of 60Hz. The use of higher elements allows a better behaviour of the response using far fewer boundary elements. Figure 4 displays the real part of the response recorded at the same receiver. The conclusions are the same.

Figure 3: Anti-plane scattered field at receiver 2. (a) $f=2\text{Hz}$; (b) $f=60\text{Hz}$.

Figure 4: Anti-plane scattered field at receiver 2 – $f=60\text{Hz}$ - Real part.
Figure 5 illustrates the results computed at 32 frequencies in the frequency range from 2 to 64Hz when the receiver is placed at position 2. Thirty boundary elements are used to model the cavity. Analysis of the results reveals the improvement of the response as the order of the boundary element increases. Notice that when the constant elements start producing bad results, the quadratic elements continue to show an excellent agreement with the closed form solution.

![Graph showing amplitude vs. frequency for different types of elements]

Figure 5: Anti-plane scattered field at receiver 2 (30 elements).

The program was subsequently tested by subjecting the cavity to SV-P waves caused by a blast source at point O. Figure 6 shows the modulus of the vertical and horizontal displacement at the receiver 2 when the source reaches the cavity with incident waves of 2 and 60Hz. These responses have the same characteristics as in the previously case. These drawings show the increased importance of using higher order elements to get better accuracy.

Figure 7 displays the real and imaginary parts of the response for the numerical and closed-form solutions when the source at position O generates a harmonically vibrating source of 60Hz. As the number of the boundary elements increases the BEM improves. The results are much better when the number of the nodes of the boundary elements increases. The plots of the quadratic elements clearly show good accuracy.

Figure 8 presents the modulus of the vertical and horizontal displacement computed in the frequency range from 2 to 64 Hz when 30 boundary elements are used to model the cavity. Again they clearly demonstrate the importance of using higher order boundary elements. Comparison shows an excellent agreement at low frequencies, and slight differences as the frequency increases. Analysis of the results obtained reveals again the improvement of the response as the order of the boundary elements increases.
Figure 6: In-plane scattered field at receiver 2
f=2Hz - (a) Horizontal displacement; (b) Vertical displacement.
 f=60Hz - (c) Horizontal displacement; (d) Vertical displacement.
Figure 7: In-plane scattered field at receiver 2 – f=60Hz
Horizontal displacement - (a) Real part; (b) Imaginary part.

Figure 8: In-plane scattered field at receiver 2 (30 elements)
(a) - Vertical displacement; (b) - Horizontal displacement.
Conclusions

A boundary element formulation for plane-strain elastodynamic problems was developed and used to evaluate problems of wave diffraction and scattering in the vicinity of cavities embedded in homogeneous media when they are illuminated by SH and SV-P line sources.

Evaluation of the singular integrals, is central to this work. Here it was achieved in closed form. Analysis of the results reveals improvement of the response as the number of the nodes in the boundary elements and the number of elements used to model the cavity increases. This behaviour was expected because the accuracy of the BEM solution depends on the ratio between the length of the incident waves and the length of the boundary elements. The use of higher elements gives a more accurate solution even with far fewer boundary elements.

References


Boundary Element Research In Europe


