Boundary element method calculation of pipe features for a fracture network
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Abstract

The boundary element method (BEM) is utilized to estimate pipe conductances for discrete fracture networks under steady-state flow. Pipes connect midpoints fracture traces within a fracture and transform a complex three-dimensional network into a simpler 1D problem. Each trace is considered to be a slit of no width. Each fracture is individually analyzed under a particular boundary condition and global fluid exchange are calculated in order to define pipe conductances. Using this formulation pipe network geometries are efficiently and precisely calculated by the BEM. Comparisons with full discretized FEM results are shown.

1 Introduction

The discrete fracture network (DFN) approach (e.g. Dershowitz et al.\(^{2}\), Long\(^{6}\), Robinson\(^{5}\)) represents a fractured rock mass as a three dimensional network of two dimensional fractures. Flow and transport in these plate networks have been solved using triangular finite elements (Miller et al.\(^{7}\), rectangular finite elements with influence functions (Herbert\(^{5}\)), and networks of orthogonal pipes (Segan & Karasaki\(^{3}\)). A simplified, approximate DFN approach has been developed by Foxford et al.\(^{4}\) This approach replaces the fractures with pipes between fracture intersections. By modeling the system using pipes rather than planar finite elements, the
number of fractures modeled on a typical workstation can be increased from 70,000 to over 300,000.

This paper presents an application of the boundary element approach to define the properties of these pipes between fracture intersections. Below, the mathematical basis of the solution is developed and results for simple preliminary examples are shown.

2 The equivalent system of pipes

In the approach of Foxford et al. a network of pipes is created from a three-dimensional discrete fracture model one fracture at a time in the following way. Each fracture plane will contain the traces of all fractures which interest the plane. By connecting the midpoints of all traces on the plane the pipes for that plane are created. Pipes for each fracture plane will link at the trace midpoints, thus creating a pipe network. The hydraulic property for each pipe for steady state flow is conductance, \( C = Q/i \), where \( Q \) is the pipe flux (m\(^3\)/s), and \( i \) is the gradient along the pipe. Pipe conductance is normally estimated empirically from fracture transmissivity \( T \) (m\(^2\)/s) and trace width \( W \) (m). In the proposed approach, the conductance is estimated as an influence factor between each trace by means of Boundary Element Method. This provides a better approximation, because the BEM can be formulated to produce conductance values which are consistent with the flow pattern within each fracture. The algorithm developed for estimating pipe conductance is developed below.

As a first approximation, conductances might be calculated by imposing a uniform head equal to 1 on trace \( i \), and equal to 0 on the remaining traces within the analyzed fracture. In this case, the conductance values \( C_{ji} \) are equal to the global discharges \( Q_{ji} \) each trace \( j \) receives due to the unit charge in \( i \). Such values would be consistent with a flow pattern generated by a constant distribution of head along each trace. However, numerical experiments have shown that this approximation is too coarse, as the heads along an individual trace frequently have large variations. The variation in head along traces can have a significant effect on flux patterns and pipe conductance. An assumed constant distribution would imply larger conductances, leading to consistent errors even in case of simple networks. Furthermore, the assumption of constant heads on traces produces a singularity in the solution where traces intersect. Thus a more sophisticated method of assigning heads to traces is necessary.

The next level approximation is to allow head to be a function of position along each trace. This can be implemented by relating the head at
each position to an external head $H$ located in the intersecting fracture, by means of a leakage factor $v$. For a generic trace $i$ on fracture $J$, the flux normal to the boundary, $\frac{\partial h}{\partial n}(x)$, is given by:

$$
\frac{\partial h}{\partial n}(x) = v_i [H_K - h(x)] \quad x \in \Gamma_i ,
$$

where $h(x)$ is the head distribution and $v_i$ the leakage factor associated with trace $i$, $H_K$ is the head value dominant in fracture $K$, whose intersection with $J$ gives $i$, and $\Gamma_i$ is the inner boundary coincident with the trace. In the following, eqn (1) is simplified by assuming that fractures have practically no width. Imposing an external head equal to 1 to fracture $i$, and 0 to the other external heads related to the remaining traces, a flow pattern is established within the fracture. In this approach, the pipe conductance can be calculated from the gradient between the traces as defined by the mean head on each trace.

For each set of boundary condition a set of linear equations can be written with unknown values of conductance $C_y$. BEM is to find the solution for each set of boundary conditions. The following section develops this solution.

3 Simulation of flow within a single fracture by means of BEM

Given a domain $\Omega$, coincident with a fracture plane of assigned uniform transmissivity, the target is the solution of the Laplace equation for steady-state flow:

$$
\nabla^2 h(x) = 0 \quad x \in \Omega ,
$$

where $h(x)$ is the hydraulic head distribution in $\Omega$ under the following condition on the outer boundary $\Gamma_E$ (i.e. the periphery of the fracture):

$$
\frac{\partial h}{\partial n}(x) = 0 \quad x \in \Gamma_E .
$$

The $nt$ traces of intersecting fractures coincide with the inner boundary $\Gamma_T$, and each must satisfy eqn (1). The $\Gamma_E$ contour is a no flow boundary unless a fracture trace coincides with the edge of a fracture, in which case eqn (1) must be satisfied. Discretizing the boundary into $nn$ nodes with $N_i$
shape functions, the BEM equation at each collocation node $i$ with coordinates $\mathbf{x}_i$ can be written:

$$c_i h_i + \sum_{j=1}^{m} \int_{\Gamma} N_j \frac{\partial h^*}{\partial n}(\mathbf{x}, \mathbf{x}_i) h_j d\Gamma - \sum_{j=1}^{m} \int_{\Gamma} N_j h^*(\mathbf{x}, \mathbf{x}_i) \frac{\partial h}{\partial n} j d\Gamma = 0,$$

(4)

with $h^*(\mathbf{x}, \mathbf{x}_i) = \ln[1/r(\mathbf{x}, \mathbf{x}_i)]/(2\pi)$, $\partial h^*/\partial n(\mathbf{x}, \mathbf{x}_i) = -[(\mathbf{x}-\mathbf{x}_i) \times \mathbf{n}]/[2\pi r(\mathbf{x}, \mathbf{x}_i)^2]$, $r(\mathbf{x}, \mathbf{x}_i)$ distance between $\mathbf{x}$ and $\mathbf{x}_i$ and $\Gamma = \Gamma_E + \Gamma_T$. Given such boundary conditions, eqn (4) can be written in compact form:

$$\begin{bmatrix} A_{E-E} & -B_{E-T} \\ A_{T-E} & -B_{T-T} \end{bmatrix} \begin{bmatrix} h_E \\ \{\partial h / \partial n\}_T \end{bmatrix} = -\begin{bmatrix} A_{E-T} \\ A_{T-T} \end{bmatrix} h_T,$$

(5)

where $h_E$ are the nodal hydraulic heads on the outer boundary $E$, $h_T$ and $\{\partial h / \partial n\}_T$ are respectively the nodal hydraulic heads and normal boundary fluxes on $T$, coincident to the traces. The coefficient matrixes $A$ and $B$ result from the numerical evaluation of the integrals in eqn (4). In the symbols $E-E$, $E-T$, $T-E$ and $T-T$, the first letter indicates the collocation boundary and the second letter the boundary where the integration is carried out. Eqn (5) is consistent with inner traces, i.e. traces not lying on an outer fracture edge. Because real fractures frequently terminate at fracture intersections, fractures can have one or more edge which coincide with traces. In this case eqn (5) becomes:

$$\begin{bmatrix} A_{E1-E1} & -B_{E1-E2} & -B_{E1-T} \\ A_{E2-E1} & -B_{E1-E1} & -B_{E2-T} \\ A_{T-E1} & -B_{T-E2} & -B_{T-T} \end{bmatrix} \begin{bmatrix} h_{E1} \\ \{\partial h / \partial n\}_{E2} \\ \{\partial h / \partial n\}_T \end{bmatrix} =$$

$$\begin{bmatrix} A_{E1-E2} \\ A_{E2-E2} \\ A_{T-E2} \end{bmatrix} \begin{bmatrix} h_{E2} \\ h_T \end{bmatrix},$$

(6)
Because the trace is considered a slit (Figure 1), as suggested by Elsworth integration is carried along a line and not along a closed boundary. As a consequence, just one node is considered. The first integral in eqn (4) is zero; the scalar product is calculated twice at each point but with two different directions of the normal. The second integral extended to $\Gamma_{\text{trace}}$ is

$$\int_{\Gamma_{\text{trace}}} q(x)h^*(x,x_i) d\Gamma,$$

where $q(x)$ is $[(\partial h/\partial n)_1 + (\partial h/\partial n)_2](x)$, with the subscripts 1 and 2 referring to the two normals in $x$. The term $c_i$ is equal to 1 for all the nodes on the traces.

Considering $q_{1T}+\{\partial h/\partial n\}_{1,T}+\{\partial h/\partial n\}_{2,T}$ as unknowns of the problem and assuming as a first approximation that $v$ is constant for all traces, by means of eqn (1) eqn (5) becomes

$$\begin{bmatrix} A_{E-E} & -B_{E-T} \\ A_{T-E} & (-B_{T-T} - I_v) \end{bmatrix} \begin{bmatrix} h_E \\ q_T \end{bmatrix} = - \begin{bmatrix} 0 \\ I \end{bmatrix} H_T.$$

In eqn (8) $H_T$ is the vector of fixed external (constant) heads (i.e. $H_T={H_1, ..., H_{K,...}, H_{LF}}$, where indexes refer to the intercepting fractures and in particular $LF$ is the fracture index related to the last trace within the analyzed fracture). In order to impose directional gradients an external head of $H_K = 1$ is imposed for the generic $i$ trace (motor trace), while in the other $j$ traces far heads are equal zero. Furthermore, assuming a distribution of leakage factors $v_T={v_1,...,v,...,v_m}$, a final equation can be written:
where the B matrix has been divided into sub-matrixes with the two subscript symbols indicating respectively which trace hosts the collocation node and on which trace the integration is performed. Sub-matrixes $A_{1,E}$, $A_{E,1}$, $A_{1,E}$, $A_{E,1}$ contain coefficients derived from numerical integration performed over the external boundary E with collocation nodes located on the nt traces. For traces coincident to edges, the same procedure can be used, starting from eqn (6) and leading to a formally similar equation.

By placing motor trace / in turn at each trace, nt equation systems like (9) can be written. These equations have the coefficient matrix in common, which can be solved for each position on the trace $i$ being solved. Once the solutions for each position of the trace $i$ are obtained, mean values of $h$ and $q$ over each trace are calculated and used to derive the pipe conductances.

The leakage factors $v$ must be determined before eqn (9) can be solved. For each trace, $v$ needs to be consistent with the ratio of transmissivity between the current fracture and the intersecting fracture. One approach for this calculation is to link it to the conductance of the pipe connections in the intersecting fracture forming the trace (Fig. 2). Consider two fractures, $J$ and $K$, whose intersection forms trace $i$. If $v_i$ is the leakage factor for $i$ in fracture $J$, $nc(i)$ are the connections to $i$ in K, fracture intersecting $J$ and determining $i$, and $HK$ is the constant dominating value of head in $K$, equal for all the traces in $K$ itself, we can write:

$$Q_i = \int_{\Gamma_i} v_i (H_K - h(x)) d\Gamma_i = \sum_{j=1}^{nc(i)} C_{ij,K} (H_K - \bar{h}_i), \quad (10)$$

where $Q_i$ is the global flux entering $i$, $C_{ij,K}$ are pipe conductances related the possible connections to $i$ in $K$ and $\bar{h}_i$ is the mean head value in $i$. Eqn (10) justifies the final assumption:
where $LT_i$ is the length of trace $i$. Eqn (11) links $v_i$ with the values of conductances in the intersecting fracture $K$. A recursive method of calculation makes it possible to estimate the factor $v_i$. The numerical approach for the solution is thus to use an estimate for $v_i$ to solve initial values of conductance, and then to use these initial values of conductance to calculate more accurate values of such a factor by means of (11). Further iteration improves the estimate. The values of conductances are particularly sensitive to the leakage factor when the traces overlap.

4 Examples of application

In this section, the method described in the previous section is applied in two examples. The first example deals with a square plane fracture with two overlapping traces divided into 4 sub-traces (ST) (Figure 3). The far hydraulic head for trace 1 is equal to 1.0, and is 0.0 for trace 2. ST 1 and ST 3 are under an external head $H$ equal 1. BEM calculation produces the expected symmetry across the diagonal and a reasonable flow pattern (arrows in Figure 3).
To better simulate the rapid variation in head and flux approaching trace extremities, the length of constant elements varies. As shown in Table 1, changing the number of elements for each sub-trace does not substantially improve the results for the mean values. A large number of elements are needed when dealing with values of the leakage factor greater than 1000 for the same example. In fact the flow around the intersection tends to infinity as the factor increases, and a finer discretization is needed to simulate the related steep change. For a leakage factor $v_1$ of 100,000, twenty constant elements per sub-trace corresponds to a stable value (i.e. increasing the number of elements further does not cause noticeable change in the global discharge values).

<table>
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<th># elem.</th>
<th>$Q_1$</th>
<th>$Q_3$</th>
<th>$H_5$ mean</th>
<th>$H_7$ mean</th>
<th>$H_6$ mean</th>
<th>$H_8$ mean</th>
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<tr>
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<td>.565</td>
<td>.777</td>
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<td>.776</td>
<td>.435</td>
<td>.224</td>
</tr>
<tr>
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<td>1.4560</td>
<td>.565</td>
<td>.776</td>
<td>.435</td>
<td>.224</td>
</tr>
</tbody>
</table>

Table 1: Sub-trace mean head values for the first example with different number of elements per sub-trace
The second example is a flow simulation of a network with 35 fractures. The network shown in Figure 4 has been produced by FracMan, the fracture mesh generator of Dershowitz et al.2 The transmissivities vary from $1 \times 10^{-07}$ to $1 \times 10^{-05}$ m$^2$/s. Two different boundary conditions are imposed. In the first, one side of fracture 16 has constant $h=1$, while fracture 8 has one side with constant $h=0$. In the second, an additional constraint, $h=0$ on one side of fracture 9 is imposed. Table 2 compares the pipe network results for example 2 against finite element simulations using triangular finite elements and the matrix fracture interaction code (MAFIC) of Miller et al.7 Two sets of results are presented for the triangular finite element solution to facilitate evaluation of the magnitude of errors due to the use of the pipe approximation as compared to errors related to coarse finite element meshes with linear basis functions. The solution based on a refined finite element mesh with quadratic basis functions is considered to be the "true" solution for this flow field. The linear basis function finite element solution uses a coarser mesh with half the elements. The pipe system conductances are obtained by running the BEM algorithm with 4 quadratic elements per external side and 12 constant element per trace. An automatic discretization has been implemented, which does not allow nodal incidences and coordinates to be stored.

Figure 4: network with 35 fractures; side in boldface of F16 is under $h=1$, sides in boldface of F8 and F9 are under $h=0$
Table 2: Comparison between flux through the model for triangular finite element (TFE) and boundary element method pipe solutions (BEM/PN) of the second example fracture network (q.: quadratic shape functions; l.: linear shape functions)

The pipe system with BEM conductances consistently provide results with errors of less than 10%, significantly smaller errors than the coarse mesh finite element solution.

In Figure 5, the head values calculated for the coarse mesh are compared to the heads calculated using the BEM conductance pipes approach. For this case, the BEM solution is not as close to the FE quadratic solutions as the linear FE solutions are, but are comparable.
Comparing the mean heads for the BEM and TFE solutions for each trace shows similar sequences (from the higher to the lower values of heads), indicating consistent flow patterns. The small boxes in figure 5 contain the values of width ($W=C/T$) for each connection. Connections with small $W$ values are suppressed.

Similar comparisons between BEM pipe and FE triangular element solutions have been carried out with larger and more complex fracture network. These comparisons confirm the results, shown in Tables 1 and 2, that the BEM pipe approach provides a reasonable alternative to simulation using triangular finite elements.

5 Conclusions

This paper describes a new approach to derive equivalent pipe conductances to facilitate simulation of three dimensional discrete feature networks. A BEM algorithm was developed to deal with the complex geometries and local heterogeneities which arise from intersecting stochastic fractures, including the effects of compound intersections.

For steady state flow, the BEM approach solves pipe network conductances which produces reasonable results when compared to a finite element simulation based on triangular elements. The BEM approach can be extended to derive properties appropriate for flow and transport simulations, and for coupled stress-flow solutions using pipe networks.

References


Segan, S. & Karasaki, K., TRINET: a flow and transport code for fracture networks, user's manual and tutorial, LBL-34834, Lawrence Berkeley Laboratory, Berkeley, CA, 1993