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# Constructal theory of organization in nature: dendritic flows, allometric laws and flight

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## Abstract

This paper draws attention to the 1996 constructal theory of the generation of geometric form in flow systems. Flow architecture can be reasoned on the basis of principle: the maximization of global performance subject to finite-size constraints. One example is the generation of tree-shaped flow patterns, as optimized paths between one point (source, sink) and an infinity of points (area, volume). The optimized tree-flow architecture accounts for allometric laws, for example, the proportionality between metabolic rate and body size raised to the power 3/4, and the proportionality between breathing and heart beating times and body size raised to the power 1/4. Another example is the proportionality between the cruising speed of flying bodies (insects, birds, airplanes) and body mass raised to the power 1/6. The "thermodynamics law" status of the constructal principle is discussed.

## 1 Constructal theory versus biomimetics

Geometric similarities and patterns abound in flow systems in engineering and in nature. For example, tree-shaped flows are everywhere, in computers, lungs, dendritic crystals, urban street patterns, and communication links. In a new book [1], I started from the design and optimization of engineered systems and developed a deterministic principle for the generation of geometric form in natural systems. In flow systems for fluid, heat, mass, electricity, goods, and traffic, better performance means improved access: minimal flow resistance, minimal travel time, minimal cost. This observation led to *constructal theory*,

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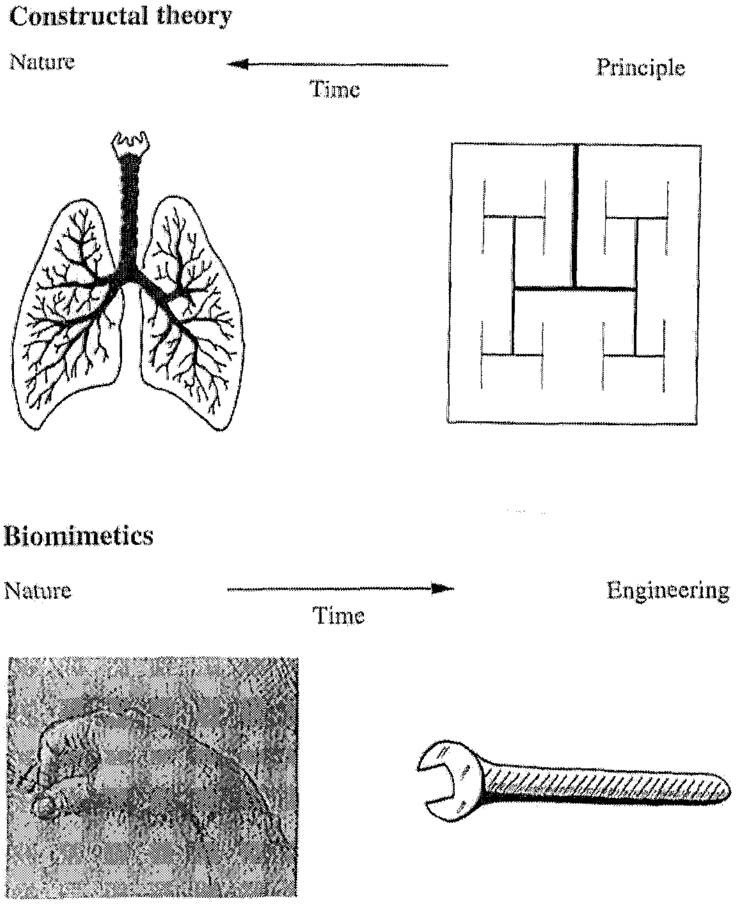

Figure 1: Proceeding against method, in time: the constructal theory of flow shape and structure in nature, versus biomimetics [1].

that is, the thought that the objective and constraints principle used in engineering also accounts for the emergence of geometry in natural flow systems.

Constructal theory was first stated in 1996, in the context of optimizing traffic [2] and the cooling of electronics [3]: "For a finite-size open system to persist in time (to survive) it must evolve in such a way that it provides easier and easier access to the currents that flow through it." The flow path was constructed in a sequence of steps that started with the smallest building block of fixed size, and continued in time with larger building blocks (constructs). The mode of transport with the highest resistivity (slow flow, diffusion, walking, and high unit cost) was placed at the smallest scales, filling completely the smallest elements. Modes of transport with successively lower resistivities (fast flow, streams, vehicles, and low cost) were placed in the larger constructs,

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where they were used to connect the area-point or volume-point flows integrated over the constituents. The geometry of each building block was optimized. The architecture that emerged was a tree—a geometric form *deduced* from a principle.

Shapeless flow (diffusion) is assigned optimally to flow with shape (streams), with the global objective of bringing the entire system to equilibrium faster. This is achieved through the development of flow structure. This is a self-standing principle, independent of the second law of thermodynamics.

The starting position occupied by theory (principle) is essential to understanding in a deterministic sense the occurrence of flow geometry in natural systems. In constructal theory deterministic progress is made by proceeding, in time, from principle (engineering) to nature, fig 1. The history of science and technology is rich with examples of the opposite flow of ideas, from nature to engineering. Today that classical method is known as biomimetics, and continues to be a very valuable tool in engineering design. An example is given in the lower part of fig 1.

## 2 Allometric laws and heat transfer theory

Living flow systems of all sizes are united by scaling (allometric) laws that correlate functional parameters with body size [4-8]. One example is the proportionality between loss of body heat and body mass raised to an exponent of order 2/3 or 3/4. The pre-1984 history of this exponent was recounted by Schmidt-Nielsen [4]. For example, Rubner theorized that the metabolic rate must be proportional to the heat loss from the body to the ambient. Because the convective heat loss is proportional to the body surface, the metabolic rate must be proportional to the length scale ($V^{1/3}$) squared, i.e. body mass or volume raised to power 2/3. This heat transfer theory is outlined in [9].

Heat transfer theory was discredited in the second half of this century by observations of birds and mammals, indicating an exponent closer to 3/4 than 2/3. Heat transfer theory was pushed aside completely by the 1977 model of West et al. [10]. See also Banavar et al. [11]. To discuss these models is not my objective. It is sufficient to note that, as in earlier optimizations of fluid tree networks in physiology and river morphology [1], the models [10, 11] were based on invoking the minimization of pumping power.

The minimization of pumping power is one of the most common manifestations of the constructal principle. If minimal pumping power is good for the performance of the global flow system (animal), then minimal loss of body heat (food intake) is also a manifestation of the constructal principle. Heat transfer cannot be dismissed.

## 3 Trees in counterflow: thermal insulation structures

Rubner's theory is about convective heat loss from the outer surface of the body to the ambient. In constructal theory [1, 12], I turned Rubner's view outside-in, and evaluated the resistance to heat transfer through the vascularized tissues under the skin.

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The main steps of this analysis are summarized in fig 2. The features of the minimal-power fluid tree [1, 13] can be deduced more succinctly by optimizing a construct consisting of a plane T-shaped junction (fig 2A). Assume right angles and Hagen-Poiseuille flow with constant properties in every tube. The stream $m_i$ encounters the flow resistance of two $L_{i+1}$ tubes in parallel, which are connected in series with one $L_i$ tube. When the resistance is minimized by fixing the total tube volume, we find the optimal diameter ratio $D_{i+1}/D_i = 2^{-1/3}$, cf. Murray's law. This result is independent of the lengths ($L_i$, $L_{i+1}$) and the relative position of the three tubes.

Next is the optimization of the lengths when the space allocated to the construct is fixed. Now geometry and relative positions are taken into account. The T-shaped construct is plane, in agreement with Zamir's [14] measurements. In fig 2A the space constraint is $2L_{i+1}L_i$ = constant. This second minimization of the flow resistance yields the optimal length ratio, $L_{i+1}/L_i = f = 2^{-1/3}$.

The issue is the flow of heat from the roots to the edges of the canopies of fluid trees such as fig 2B. This represents heat lost by an animal through the volume situated under its skin. The trees of blood vessels are one optimized geometric feature. Another is the *superposition* of the arterial and venous trees, so closely and regularly that tube i of one tree is in counterflow with tube i of the other, fig 2C. Superpositions of warm and cold streams in counterflow are

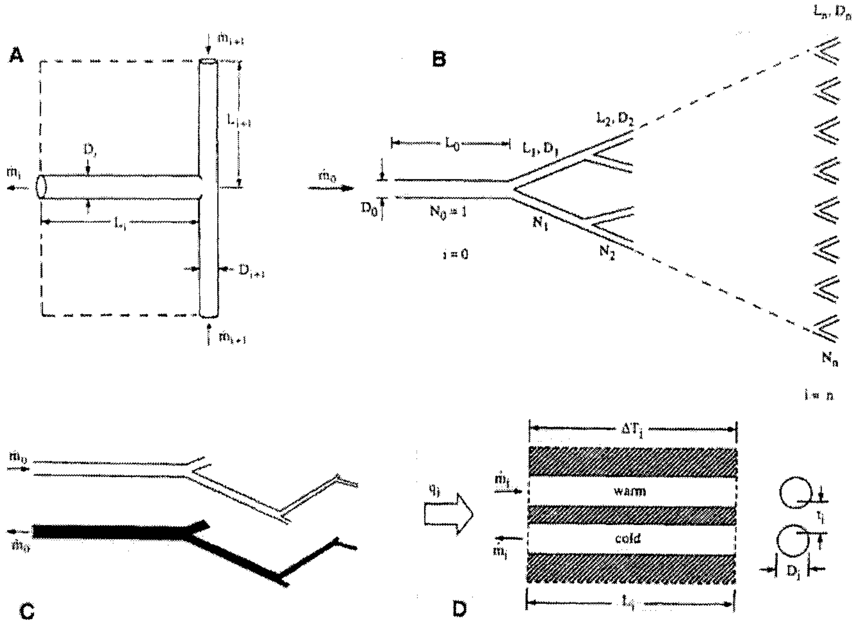

Figure 2: The construction of the tree of convective heat currents: (A) the constrained optimization of the geometry of a T-shaped construct; (B) the stretched tree of optimized constructs; (C) the superposition of two identical trees oriented in counterflow, and (D) the convective heat flow along a pair of tubes in counterflow [1, 12].

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arrangements optimized for thermal insulation in the longitudinal direction [9, 15].

The counterflow formed by two tubes of level i is shown in the detail of fig 2D. The arterial stream is warmer than the venous stream: heat flows transversally, from stream to stream. Because the enthalpy of the warmer stream is greater than that of the colder stream, the counterflow convects longitudinally the energy current $q_i = \dot{m}_i c_p \Delta T_{t,i}$, where $c_p$ is the blood specific heat, and $\Delta T_{t,i}$ is the stream-to-stream temperature difference at level i. It was shown in 1979 in the field of heat transfer [15], and in 1985 in bioengineering [16] that the counterflow convects an energy current proportional to $\Delta T_i/L_i$,

$$q_i = \frac{\dot{m}_i^2 c_p^2}{h_i p_i} \frac{\Delta T_i}{L_I} \tag{1}$$

Here $h_i$ and $p_i$ are the overall stream-to-stream heat transfer coefficient and the perimeter of contact between the two streams. In the case of blood counterflow, the stream-to-stream thermal resistance $h_i^{-1}$ is the sum of the resistance through the fluid in the duct (~ $D_i/k_f$, where $k_f$ is the fluid thermal conductivity), plus the resistance through the solid tissue that separates two tubes (~ $t_i/k$, where k is the tissue thermal conductivity). Even when the tubes touch, $t_i$ is of the same order as $D_i$. Furthermore, $k_f \sim k$, $h_i \sim k/D_i$, and eqn (1) becomes

$$\Delta T_i \sim q_i L_i k / \left(\dot{m}_i^2 c_p^2\right) \tag{2}$$

## 4 Body heat loss vs. body size

The double-tree structure constitutes a single tree of convective heat streams with zero net mass flow. The convective tree stretches from the core temperature of the animal (at i = 0) to the skin temperature (i = n). The many counterflows of the double tree sustain the overall temperature difference

$$\Delta T = \sum_{i=0}^{n} \Delta T_i \sim \frac{q_0 k}{\dot{m}_0^2 c_p^2} \sum_{i=0}^{n} N_i L_i \tag{3}$$

where we used the continuity relations for fluid flow ($N_i \dot{m}_i = \dot{m}_0$, constant) and heat flow ($N_i q_i = q_0$, constant). Recalling the $L_{i+1}/L_i$ constant, we substitute $L_i = L_0 f^i$, $L_n = L_0 f^n$ and $N_i = 2^i$ into eqn (3),

$$q_0 \sim \left(\frac{q_0}{\dot{m}_0}\right)^2 \frac{k L_n f^{-n} \left[(2f)^{n+1} - 1\right]}{c_p^2 \Delta T (2f - 1)} \tag{4}$$

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On the right-hand side we see quantities that are constant, and quantities that depend on n (the number of construction steps). The ratio $q_0/m_0$ is independent of body size (n) because both $q_0$ and $m_0$ are proportional to the metabolic rate.

The volume occupied by the double tree is estimated by regarding the stretched tree as a cone in fig 2B. The cone base (i = n) has an area of size $N_n L_n^2 \sim 2^n L_n^2$. The cone height is of the same order as $L_0 + L_1 + \ldots + L_n = L_0(1 - f^{n+1})/(1 - f)$. In conclusion, the volume scale is

$$V \sim L_n^2 \left(\frac{2}{f}\right)^n \frac{1 - f^{n+1}}{1 - f} \tag{5}$$

The relation between metabolic rate and volume is obtained by eliminating n between eqns (4) and (5). The result is visible if we assume that n is sufficiently large so that $(2f)^{n+1} >> 1$ in eqn (4) and $f^{n+1} << 1$ in eqn (5). In this limit $q_0$ is proportional to $2^n$, and V is proportional to $(2/f)^n$, hence

$$q_0 = (\text{constant})\ V^{3/4} \tag{6}$$

In conclusion, if the tree counterflow is the dominant thermal resistance, the proportionality between metabolic rate and body size raised to the power 3/4 is predictable from heat transfer theory. It can be verified numerically that eqn (6) is also accurate for small n: when n increases from 1 to 2, $q_0$ increases as $V^{0.744}$.

## 5 Breathing and heartbeating times vs. body size

The preceding theory also accounts for the proportionality between breathing (or heartbeating) time and body size raised to the power 1/4 [1]. In my third constructal paper [17], I found that the pumping power required by the heart, or the thorax for breathing, is minimal if the flow is intermittent (on & off), and if the 'on' time interval ($t_1$) is of the same order of magnitude as the off time interval ($t_2$). The optimal time scale ($t_{1,2} \sim t$) is

$$t \sim \left(A D^{1/2}\, \Delta C / \dot{m}\right)^2 \tag{7}$$

where A, D, ΔC and $\dot{m}$ are the total internal contact area of all the tubes of the tree, mass diffusivity, concentration difference that drives the mass transfer process, and total mass flow rate of the tree (air, blood). The flow rate $\dot{m}$ is proportional to the metabolic rate of the animal. The minimization of pumping power, and the optimal temporal structure (rhythm) represented by eqn (7) are demanded by the constructal principle.

To predict t as a function of body mass (M) we need expressions for $\dot{m}(M)$ and A(M) to substitute in eqn (7). From the analysis of the optimized tree of convective currents, we obtained eqn (6) or $\dot{m} \sim M^{3/4}$. For the derivation of the relation A(M), consider the following geometric argument. The thickness of the tissue penetrated by mass diffusion during the breathing or heart beating time t is proportional to $t^{1/2}$. The body volume (or mass) of the tissue penetrated

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by mass diffusion during this time obeys the proportionality $M \sim At^{1/2}$. Eliminating t between $M \sim At^{1/2}$ and $t \sim (A/m)^2$, and using $m \sim M^{3/4}$, we obtain

$$A \sim M^{7/8} \tag{8}$$

This trend is in agreement with many measurements reported in physiology (e.g., ref. [4], p. 113). Finally, by substituting $m \sim M^{3/4}$ and $A \sim M^{7/8}$ into $t \sim (A/m)^2$, we anticipate that the optimized time intervals should vary as

$$t \sim M^{1/4} \tag{9}$$

This allometric law is supported convincingly by the physiology literature [4-6].

## 6 Comparison with the real world

In summary, the heat transfer theory reviewed in sections 2-5 represents the combination of (i) the optimization of fluid tree architecture for minimal pumping power, and (ii) the maximization of thermal insulation by placing warm and cold trees in counterflow. Putting (i) and (ii) together into a body heat-loss theory of animal design is the contribution of constructal theory.

The convective thermal resistance posed by the trees in counterflow ($R_1$, fig 3) resides inside the animal. This runs in parallel with a second internal resistance ($R_2$) associated with the conductive heat leak through the solid tissue. The conductive resistance $R_2$ is proportional to the body thickness scale $V^{1/3}$ divided by the body surface $V^{2/3}$, hence $R_2 \sim V^{-1/3}$. The tree resistance $R_1$ is proportional to $V^{-3/4}$. The ratio $R_2/R_1 \sim V^{5/12}$ shows that $R_2$ becomes progressively weaker (i.e., the preferred path) as the body size decreases. In that limit the exponent in the power law between heat loss and body size becomes 1/3. I wrote in 2000 [1] that "from heat transfer theory alone we should expect a gradual decrease in the power-law exponent as the body size decreases. At the other end, 3/4 is the asymptotic value of the exponent for large body sizes." There is no unique value for the exponent in the power law (6).

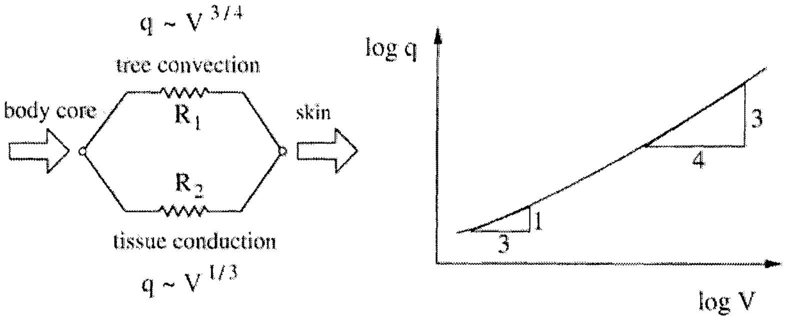

Figure 3: The exponent in the metabolic rate allometric relation increases as the body size increases [1, 12].

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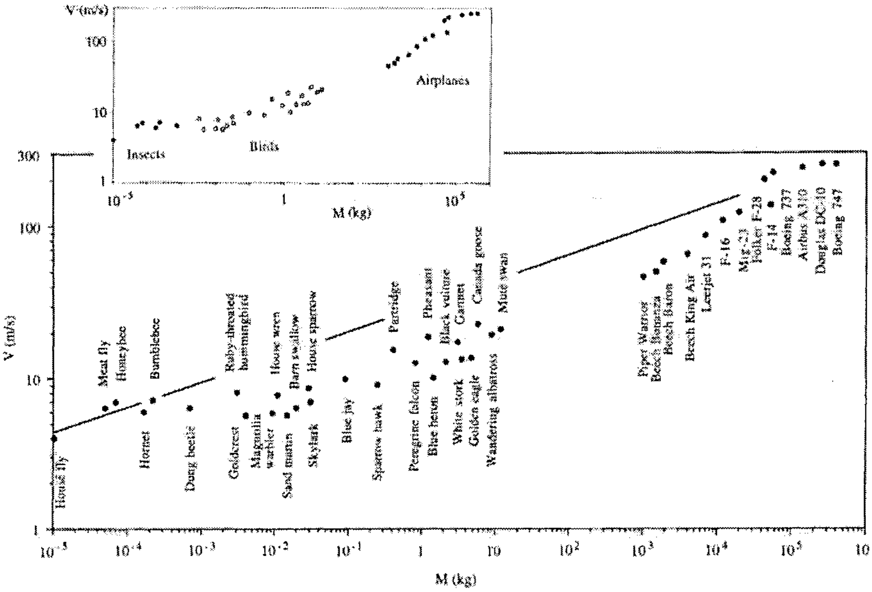

Figure 4: The flying speeds of insects, birds and airplanes, and the theoretical speed for optimal therodynamic performance [1].

These predictions were confirmed in their entirety one year later by Dodds et al. [18], who reported an exhaustive re-examination of the large amount of empirical data on metabolic rate and body size. Dodds et al. [18] concluded that there is little evidence in support of rejecting Rubner's heat transfer theory [an exponent of 2/3 instead of 3/4 in eqn (6)], which they consider the "null hypothesis". Dodds et al. [18] found systematic evidence that for mammals the exponent in eqn (6) depends on body size, and increases as the body size increases. Furthermore, Dodds et al. [18] showed that recent (non-heat transfer) attempts to derive the 3/4 exponent [10, 11] are mathematically incorrect and do not prove their stated conclusions.

I use this opportunity to stress that constructal theory [2, 3, 9, 17] was published before the model proposed by West et al. [10]. The latter proposed a model of an assumed tree of tubes for fluid flow. They acknowledged only three of the assumptions that they made, which I quote: (i) the flow structure is shaped as a tree, (ii) the final branch of the tree is a size-invariant unit, and (iii) the energy required to distribute resources is minimized. These three features were present already in constructal theory, but there is an important difference. In constructal theory, (i) and (ii) were not assumed. They were deduced from (iii), which is the constructal law discussed in section 1.

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## 7 Constructal law: maximized access for currents

The advances reviewed in this article originated from the theory that it is natural for currents to construct for themselves paths of least resistance. This thought led to the designed tree-shaped flows with which constructal theory began. Minimal resistance also means minimal irreversibility, or minimal exergy destruction in systems with internal currents. In this way the constructal principle covers every flow system that strives to achieve better performance—(a) natural inanimate systems, such as the river basin and the cracking mud, (b) natural animate systems, such as all the animals and their respiratory and circulatory networks, and (c) engineered systems. The latter are our own extensions, and they improve our performance and chance of survival as individuals and societies.

Systems (c) are special cases that belong in the much larger class (b), and, for this reason, the constructal principle means that to engineer is natural, or that nature engineers. This point is pressed with vigor by the data assembled in fig 4. The theoretical flying speed for minimal fuel (or food) consumption is proportional to the flying mass raised to the power 1/6, namely

$$V_{opt} \sim \rho_a^{-1/2} \, \rho_b^{1/3} \, g^{1/2} \, M^{1/6} \tag{10}$$

where $\rho_a$, $\rho_b$, g and M are the air density, body density, gravitational acceleration and body mass. When the speed is $V_{opt}$ the exergy destroyed by flow drag and the repeated lifting of the flying body is minimal. This theoretical result is known in biology and engineering [19, 20], and was also derived based on constructal theory (ref. [1], pp. 234-242). The flying data on insects, birds and aircraft support it convincingly.

Classes (a) and (b) are one and the same from the constructal point of view. The principle proclaims the existence of optimal and many near-optimal designs of flow architectures under constraints. In biology this goal is known as the "fittest" (or its equivalent "the flow structure that survives"). The constructal law places this goal on a universal theoretical foundation. All the flow (nonequilibrium) systems that have the ability to morph their configurations under constraints, will progress in steps of geometric form toward better performance—conglomerates of forms that flow more easily. They survive because they change, i.e., they project themselves into the future by flowing through beneficially altered configurations.

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