A novel method for the solution of the three-dimensional dynamic crack problems

E. I. Shifrin\textsuperscript{1} & A. Staroselsky\textsuperscript{2}
\textsuperscript{1}Moscow Aviation Technology University, Moscow, Russia
\textsuperscript{2}United Technologies Research Center, Hartford, Connecticut, USA

Abstract

We present a novel, efficient and robust numerical method to handle spatial problems for practically arbitrarily shaped plane cracks or crack systems under static or dynamic loading. A two-parametric method previously used for static analysis is generalized for dynamic shear problems. A system of integral-differential equations for a displacement jump in the crack plane was derived for generalized loading conditions and solved by this method. In this paper, values of Stress Intensity Factors (SIF) are obtained for a penny-shaped crack, subjected to either harmonic or impact loads. The results are shown to be in good accord with known analytical results for static loading and numerical results of other investigators.

Introduction

Dynamic response of elastic media subject to transient excitations is extremely important for two reasons. First, interaction of a planar crack in three-dimensional solids with time-harmonic elastic waves is a foundation for developing non-destructive methods of crack detection. In particular, the scattering of non-stationary time-dependent loads/waves by cracks is a point of practical and theoretical interest. The second area of engineering activities where the response of cracks to dynamic loading must be considered, is damage tolerant and life cycle design, which is now widely used in high technology industrial applications. This practice requires the prediction of crack behavior subject to dynamic loading.
Both wave reflection and crack-tip field analysis require the solution of the Navier equations [1] for three-dimensional space. Spatial crack analysis presents a real challenge even for static problems. Consequently, a limited number of complete solutions is known for arbitrarily shaped cracks. Difficulties arise when dynamic problems (governing equations contain the inertia term $\rho \frac{\partial^2 u}{\partial t^2}$) are to be considered. The finite element method has been extensively used for solving fracture mechanics problems [2]. However, it is not effective for problems with singularities (a fine mesh is needed near the crack front) and for problems in an infinite domain. The boundary integral equation method (BIE) [3] has also been widely used in continuum mechanics. It reduces the order of the problem, which is well suited for problems with singularities [4], however, the system matrix is full and unsymmetric so BIE is also computationally expensive. Using BIE and semi-analytical methods the case of crack interaction with longitudinal waves has been analyzed by Mal [5], Martin [6], and Budreck and Achenbach [7]. Recently, there have been a number of works (for example, Annigeri and Cleary [4]) aimed at developing a hybrid method combining advantages of both numerical schemes, however, there is a need in an efficient and robust numerical method to handle spatial problems for nearly arbitrarily shaped plane cracks or crack systems under static or dynamic loading.

We present a novel two-parametric method that gives us an opportunity to use a few elements of basic functions. This approach can be applied for solving problems with arbitrarily shaped planar cracks. In this paper we apply this method to obtain a complete solution for the scattering of shear harmonic loads by a planar crack in three-dimensional space. We consider the case of wave scattering by a penny-shaped crack as a model example to verify our results against results of other investigators, particularly Jia, Shippy and Rizzo [8] who gave the BIE solution for a penny-shaped crack for both a longitudinal and a shear harmonic wave.

The plan of this paper is as follows. In the next section we deduce the governing system of integral-differential equations and state the physical problem. We will also show that the solution for an arbitrary incident wave can be represented as the superposition of two independent problems: shear and longitudinal. Next we briefly describe the major idea of the numerical method and derive the principal relationships. In the following section we evaluate the applicability of this method for the analysis of harmonic shear wave interaction with a penny-shaped crack. Our calculation results show that the two-parametric method is extremely efficient and accurate for such a class of problems. We close in the last section with some final remarks. It is important to note that due to the paper size limitation, we have omitted most of technical details and derivations. We just attempt to explain the main idea and the spirit of the numerical method and demonstrate its applicability.
Fundamental equations

The three-dimensional steady-state harmonic displacement wave projects normally to the crack $G$ in plane $x_3 = 0$. This problem can be reduced to that of a crack whose upper and lower surfaces are forced with equal effort in opposite directions, such that they relieve the external stresses defined by the incident wave. It is assumed that crack surfaces do not interact with each other and Sommerfeld’s radiation conditions are assumed at infinity. For the initial problem, the stresses applied to the upper side of the crack are equal to $t e^{\pi(\omega, t)}$, where $t = (t_1, t_2, t_3)$, $t_1 = \mu \omega U_1 / \rho$, $t_2 = \mu \omega U_2 / \rho$, $t_3 = (\lambda + 2 \mu) \omega U_3 / \rho$. Here $c_\rho = (\rho / \rho)^2$ is the velocity of a transverse wave, $c_d = (\lambda + 2 \mu)^2 / \rho$ is the velocity of a dilatational wave, $\lambda$ and $\mu$ are Lame’s elastic constants; $\rho$ is the density. If applied loads have the term $t e^{\pi(\omega, t)}$, the displacements are also harmonically dependent on time as $u e^{\pi(\omega, t)}$, where $u = (u_1, u_2, u_3)$ is a vector of displacement amplitudes.

The Navier equations for the amplitudes have the form:

$$\begin{align*}
(\lambda + \mu) u_{ij,ij} + \mu u_{ij,jj} &= -\rho \omega^2 u_i \\
\end{align*}$$

(1)

Application of the Fourier transform reduces the problem to the solution of a system of integral equations in the frequency domain. Let $x = (x_1, x_2)$. Denote by $f(\xi), \xi = (\xi_1, \xi_2)$ the Fourier transformation of the function $f(x)$:

$$\hat{f}(\xi) = \int_{R^2} f(x) e^{i(x, \xi)} dx, \quad (x, \xi) = x_1 \xi_1 + x_2 \xi_2$$

After taking the second derivative and making elementary but cumbersome transformations, the equations reduce to the form Itou [9]:

$$\left( \frac{d^2}{dx_3^2} - n_1^2 \right) \left( \frac{d^2}{dx_3^2} - n_2^2 \right) \hat{u}_i = 0. \quad i = 1 - 3$$

(2)

We use the following notations: $n_1^2 = \xi_1^2 - \alpha^2; \beta = \sqrt{\rho / \mu} \omega, \alpha = \sqrt{\rho / (\lambda + 2 \mu) \omega}, n_2^2 = \xi_2^2 - \beta^2, \xi^2 = \xi_1^2 + \xi_2^2$. Further, let us choose the branches for $n_1$ and $n_2$. Suppose that $\sqrt{s}$ is a positive number if $s > 0$, and $\sqrt{s} = -i \sqrt{|s|}$, if $s < 0$.

From (2) it follows that any solution has the form:

$$\hat{u}_i = A_i e^{\alpha x_3} + B_i e^{\beta x_3} + C_i e^{\alpha x_3} + D_i e^{\beta x_3}.$$  

(3)

We consider the solutions in upper and lower semi-space separately and use boundary conditions in the far field and crack plane to match them. As a result, we obtain the general operator expression. According to the problem
statement $\sigma_{i3}^+ = \sigma_{i3}^- = \sigma_{i3}$ is in the crack plane $x_3 = 0$. The displacement jump in this “matching” plane is equal to zero everywhere out of the crack: $[u_i] = 0$, $x \notin G$. Finally, after some algebraic transformations, which we skip for brevity sake, we obtain

$$\frac{\sigma_{33}}{\mu} = M(\xi) [u_3] \quad \text{and} \quad \frac{\sigma_{j3}}{\mu} = S_j [u_j] + T [u_3 - j]; \quad (j = 1, 2) \quad (4)$$

The brackets indicate a jump of the value put in them in the plane $x_3 = 0$ ($[u_i] = u_i^- - u_i^+$) and $M(\xi) = (2\xi^2 - \beta^2)^2 - 4n_1n_2\xi^2)/(2\beta^2n_1$).

$$S_j(\xi) = (4\xi^2n_2^2 - \beta^2(n_2^2 - \xi^2) - 4n_1n_2)/(2\beta^2n_2), \quad T(\xi) = \xi_1\xi_2(4n_2^2 + \beta^2 - 4n_1n_2)/(2\beta^2n_2).$$

The traction $t_1$, acting on the surface $x_3 = 0$ which bounds the upper semi-space, differs from $\sigma_{i3}$ only by sign. This means that (4) defines the system of pseudo-differential equations connecting the amplitudes of displacement jumps in a crack plane with amplitudes of efforts.

$$A_\beta [u] = t/\mu \quad (5)$$

Here, $[u] = ([u_1], [u_2], [u_3])$, and $A_\beta$ is the matrix pseudo-differential operator with symbol:

$$A_\beta(\xi) = \begin{vmatrix} K_{11}(\xi) & K_{12}(\xi) & 0 \\ K_{12}(\xi) & K_{22}(\xi) & 0 \\ 0 & 0 & K_{33}(\xi) \end{vmatrix}$$

where $K_{jj}(\xi) = -S_j(\xi), \quad (j = 1, 2)$ and $K_{33}(\xi) = -M(\xi), \quad K_{12}(\xi) = K_{21}(\xi) = -T(\xi)$.

It can be seen from (5), that the dynamic problem of plane cracks in an elastic medium under an arbitrary load is divided into two independent problems: a normal separation and a shear. The normal separation component is coincident with the Itou equation [10]. Thus, the boundary problem for a plane crack under an arbitrary load can be formulated as follows:

$$P_G A_\beta [u] = t/\mu \quad \begin{cases} [u] = 0 & \text{if } x \notin G \end{cases} \quad (6)$$

Here $P_G$ is the restriction of the operator $A$ to the crack domain $G$.

The numerical method

In this section we briefly discuss the major idea and the general approach of the two-parameter method [11]. We show the major steps without paying too much attention to strict mathematical conditions and proofs which are given in [11], and then apply it to the governing equations (6).
Let $A$ be a linear operator which maps a separable Hilbert space $H$ to a conjugate space $H'$ isomorphically. We solve the equation:

$$Au = f; \quad f \in H', \quad u \in H. \quad (7)$$

We are looking for the approximate solution in the form $u_n = \sum_{i=1}^{N} c_i e_i$, where $e_1, e_2, \ldots, e_k, \ldots$ is a complete system of functions in $H_A$. Let $\phi_1, \ldots, \phi_p \ldots$ be another orthonormal basis in $H_A$. In order to use the Galerkin approach, the equation (7) has to be multiplied by weight functions $\phi_k$, and the approximate solution could be found by minimizing the residual:

$$S^2_N(u) = \sum_{k=1}^{N} \left( \sum_{i=1}^{N} (c_i (A e_i, \phi_k) - (f, \phi_k)) \right) \left( \sum_{i=1}^{N} (c_i (A e_i, \phi_k) - (f, \phi_k)) \right). \quad (8)$$

Here and below, the line on top of an expression means the complex conjugate. If $A$ is a real operator, the product in the above expression is simply a square.

For some operators, calculations of the inner product $(A e_i, e_j)$ necessary for the Bubnov-Galerkin method are very difficult. This is due to difficulties in calculating $A e_i$. The idea of the method modification is as follows: by applying the properties of adjoint linear operators $(A e, \phi) = (e, A^* \phi)$ we reduce the problem to the calculation of operator from the specially designed weight functions. As will be shown below, it is possible to build a basis $\phi_1, \ldots, \phi_p$ such that it is rather easy to calculate $A^* \phi_j$ (even analytically for a number of problems) and subsequently an inner product $(e_i, A^* \phi_j)$. Consequently, computational time for its calculation is negligible. Thus, the method used in this problem contains the following: two basis systems of functions $e_i(x), \phi_i(x)$ are chosen. The basis $e_i(x)$ takes into account the known behavior of the solution in such a way that it can be approximated with comparatively few basis elements. The basis $\phi_i(x)$ is chosen as orthonormal in $L_2(G)$, and in such a way that $A^* \phi_j(x)$ are calculated easily; $(A^* \phi_j \text{ is the conjugate operator to } A)$.

Unknown coefficients $c_i$ are defined from the condition of minimizing $S^2_N(u)$. Because matrix operator $A$ is complex, all unknown coefficients $c_i$ are complex as well. The condition of minimization, $\frac{\partial S^2_N(u)}{\partial c_i} = 0$, together with the equality $(A e, \phi) = (e, A^* \phi)$ reduces the problem to the linear system:

$$\sum_{i=1}^{N} c_i \sum_{k=1}^{N} (e_i, A^* \phi_k)(e_j, A^* \phi_k) = \sum_{k=1}^{N} (e_j, A^* \phi_k)(f, \phi_k); \quad (j = 1..N). \quad (9)$$

Now we have to define the "weight" function. Let us take $\phi_i(x)$ as a system of vector functions, obtained by orthogonalization in $L_2(G)$ of the following system:

$$\psi^t = (\psi^t_A(\bar{x}), 0); \quad \psi_b = (0, \psi^t_A(\bar{x})); \quad \text{where } \alpha = (\alpha_1, \alpha_2, \alpha_3), \gamma \geq \frac{1}{2}.$$
\[ \psi_2(\overline{x}) = \left( \alpha_2^2 - |\overline{x} - \alpha_m|^2 \right)^{\gamma}, \quad \text{when } d < 1; \]

\[ \psi_2(\overline{x}) = 0, \quad \text{when } d \geq 1; \]

Here \( \alpha_m = (\alpha_1, \alpha_2) \) and \( d = \frac{|\overline{x} - \alpha_m|^2}{\alpha_3^2} \). The circles have radii equal to \( \alpha_3 \) and centers at points \( \alpha_m \), belonging in \( G \). In other words, we cover the crack surface by overlapping circles, define finite “bell” type basis functions on each of them and then calculate \( A^m_\phi_k \). After that, we integrate the product \( (e_j, A^* \phi_k) \) over the crack area \( \tilde{G} \) and find the solution of the linear system (9). As noted above, the functions \( \phi_k \) are a linear combination of the functions \( \phi_k \) and by linearity of the operator \( A \), we calculate \( A\phi_k \) easily. Thus, the numerical procedure of the solution to this problem is nearly complete.

**Results of computations**

In this paper, values of Stress Intensity Factors are obtained for a penny-shaped crack of a unit radius and treated with harmonic shear loading. Here, the system of basis functions \( e_i(x) \) has been chosen in form \( e_i(x) = \sqrt{1 - r^2} g_i(x) \). Without loss of generality, it was assumed that the load has the form \( f(x) = (\text{Const, } 0) \). The solution possesses some features of symmetry in this case. Taking them into account, we choose vector functions \( g_i \) in the form:

\[ g_i(x) = (\cos(j\pi r) \cos(2m\theta), 0) \]

\[ g_i(x) = (0, \cos(k\pi r) \sin(2n\theta)), \]

where \( \theta \) is the angle in the polar frame and \( j, m, k, n \) are whole numbers. The solution of the system (9) gives us values of the coefficients \( c_i \), whereby we determine the jumps of displacement in the crack region \([u_1]\) and \([u_2]\). After that, we have the opportunity to compute all of the important characteristics of the solution. The vector of displacement jumps is approximated by the sum \( \sum_{i=1}^{\infty} c_i e_i \). Let us denote \( \sum_{i=1}^{\infty} c_i g_i = [u^0] = ([u^0_n], [u^0_r]) \). It immediately follows that \( [u_j] = \sqrt{1 - r^2} [u^0_j], (j = 1, 2) \). From here and by definition of SIF by J. Rice [1], it follows that:

\[ K_2 = \frac{\mu \sqrt{\pi} [u^0_n]}{2(1 - \nu)} \quad K_3 = \frac{\mu \sqrt{\pi} [u^0_r]}{2} \]

where \([u^0_n]\), \([u^0_r]\) are projections of vector \([u^0]\) to the normal and to the tangent, respectively, of the crack contour. Their values are taken in the considered point of the contour.

It is obvious that with increasing the wave number \( \beta \), it is necessary to increase the number of basis elements \( e_i(x) \) to obtain a good approximation. For values of \( \beta \leq 5.5 \), the number of basis elements was brought
Figure 1: Normalized SIF (a) $k_2 = |K_{II}(\beta)|/K_{II}^{\text{static}}$ and (b) $k_3 = |K_{III}(\beta)|/K_{III}^{\text{static}}$ vs. wave number $\beta$ for harmonic shear loading.

up to 14. The results of these calculations were stable for a varied number of elements of the basis $e_i(x)$. The number of elements of the second basis has been varied from 50 to 70. For this purpose 25-35 functions of the type $\psi_\gamma(x)$ were used. The value of index $\gamma$ is varied from 2 to 4 in the functions $\psi_\gamma(x)$. The obtained results were close to each other. The error of the computation when $\beta = 0$ with respect to a known exact solution was less than 3 percent. The graphs of normalized stress intensity factors $K_2^*(\beta) = |K_2(\beta)|/K_2(0)$ and $K_3^*(\beta) = |K_3(\beta)|/K_3(0)$ for values of Poisson ratios $\nu = 0.1$, $\nu = 0.3$, and $\nu = 0.5$ are given in Fig. 1a and Fig. 1b. The values of $K_2^*(\beta)$ are presented in the point $\theta = 0$, and $K_3^*(\beta)$ are presented in the point $\theta = \pi/2$. Despite the fact that for $\nu = 0.5$ the P-wave velocity is infinite, Equation (6) is still valid. As we can see from these figures, increasing $\nu$ to 0.5 decreases the maximal values of $K_{2\max}^*$ and $K_{3\max}^*$ in contradiction to the case of a normal incident P-wave. As shown by Kaptsov and Shifrin [12] increasing $\nu$ to 0.5 causes a significant increase of the maximal value of the SIF for longitudinal waves. The curves corresponding to $\nu = 0.3$ are in good accord with results obtained in [8] for $\nu = 0.25$. The second maximum of $|K_3|$ is smaller than the first one for all values of Poisson ratio. $|K_2(\beta)|$ monotonically decreases with the wave number at least up to $\beta = 5.5$. We found that the second maximum of function $K_3^*(\beta)$ for $\nu = 0.3$ is far less than the first.

Concluding remarks

A new efficient numerical method is presented and applied for the analysis of three-dimensional crack problems. We modify the traditional Galerkin approach by introducing two different systems of basis functions. The first
basis has a proper crack tip asymptotic behavior and we approximate the solution with a few elements. To avoid computational difficulties we determine the second basis based on numerical ease of use of the adjoint operator on these functions. We use the second system as weight functions in the Galerkin method. By applying the properties of adjoint linear operators, we reduce the problem to the approximation of the solution by several proper basis functions. The calculation of the operator from the specially designed weight functions is also easily completed.

We show solutions for a crack loaded by harmonic shearing. The results obtained for harmonic loadings allow us to construct solutions for the general case of non-stationary dynamic loadings.

References