The bridged external circular crack

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Abstract

This paper examines the axisymmetric problem of an external circular crack, which is located in a fibre-reinforced elastic solid. In contrast to the classical problem where the surfaces of the crack region are traction free, this paper considers the situation in which fibres exhibit continuity across the external crack thereby inducing a displacement dependent traction boundary condition. The influence of the bridging action on the crack opening mode stress intensity factor at the crack tip, created by axisymmetric loading of a doublet of concentrated forces, is examined.

1 Introduction

Unidirectional fibre-reinforced composites consist of a matrix reinforced with a random or regular network of closely spaced aligned fibres. The elastic stress analysis of these composites can be performed by idealizing the material as a transversely isotropic elastic medium. The overall elasticity parameters associated with the transversely isotropic elastic idealization can be estimated by recourse to a theory of mixtures. Hashin and Rosen [1] and Hill [2] give examples of such estimates. Christensen [3] gives a comprehensive review of studies in this area. The study of fracture processes in such unidirectional fibre reinforced composites is of fundamental importance to their engineering design. The articles by Kelly [4] and Sih [5] clearly illustrate the various phenomena, such as crack formation, fibre pullout, crack bridging, matrix microcracking, matrix yielding, fibre debonding, delamination, etc., which lead to fracture of unidirectionally reinforced composites. In particular, they emphasize the possibility of crack or flaw bridging in unidirectional fibre reinforced composites that could occur as a result of intact fibre continuity across the faces of the void. The presence of fibre continuity across a crack results in displacement dependent traction boundary conditions in the region of the crack. The analysis of such mixed boundary problems associated with non-classical traction constraints has been reported in the literature on fracture mechanics, without specific relevance to modelling of flaw bridging in fibre reinforced materials (see, e.g., Atkinson [6]). Selvadurai [7,8] and Selvadurai and Au [9] examined crack bridging in uni-directional fibre reinforced materials by
considering analytical and computational approaches. References to further articles that deal with the influence of crack bridging action in fibre and inclusion-reinforced materials are given by Stang [10], Rose [11], McCartney [12], Budiansky and Amazigo [13] and Selvadurai et al. [14].

The present paper considers the problem of an external crack that is located in a unidirectional fibre reinforced material. It is assumed that fibres exhibit continuity across the surfaces of the external crack. The plane of symmetry of the intact circular region of the unidirectional fibre reinforced solid (of radius a) is assumed to be normal to the fibre direction. In order to model the elasticity of the bridging action, the length of the bridging fibres are assumed to be finite and constant (length 2l) across the external region. Although there is little published information that establishes probable limits for the ratio (a/l) (see, e.g. Selvadurai [7]) it is evident that processes such as delaminations of the stretched fibres at the faces of the crack can increase the “effective length” of the bridging fibres even though the thickness of the crack can be considerably smaller than the dimensions of the intact region. Finally, it is assumed that the set of continuous bridging fibres can be represented as a series of one-dimensional elements with independent unilateral mechanical action. This latter assumption closely follows the “Winkler Model” approach adopted by Barenblatt [15] and Goodier and Kanninen [16] and others in modelling finite and non-linear cohesive forces at the crack tip. Admittedly, the continuum modelling approach that results from this idealization does not take into account the fine structure of fibre configuration at the tip of a reinforced solid. Nonetheless, it provides a useful analogue for the examination of global effects of cohesive forces, crack bridging, fibre yielding and fibre fracture on the displacements and stresses in the sense of a continuum analysis which in turn can be used to provide estimates for global conditions at the crack tip. The mathematical analysis of the bridged external crack problem is reduced to a mixed boundary value problem associated with a halfspace region. To provide finite estimates for the stress intensity factors at the crack tip, the bridged external crack is loaded by a doublet of concentrated forces that are located at finite distances from the intact circular region. The Fredholm integral equations that result from the mathematical formulations are numerically solved to obtain results of engineering interest.

2 Basic equations

Considering the class of axisymmetric problem related to a transversely isotropic, elastic medium in which the z axis of the cylindrical polar coordinate system \((r, \theta, z)\) coincides with the axis of material symmetry, it can be shown that (see, e.g., Elliott [17]) the displacement and stress fields can be expressed in terms of two functions \(\phi_i(r, z)\) \((i = 1, 2)\) which are solutions of

\[
\left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right] \phi_i(r, z) = 0 ,
\]

where \(z_i = z/\sqrt{v_i}\) and \(v_i\) are the roots of the equation

\[
c_{11}c_{44}v_i^2 + \left[c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}\right]v_i + c_{33}c_{44} = 0 .
\]

and \(c_{ij}\) are the elastic constants of the transversely isotropic material. We note that, in general, \(c_{ij} = c_{ij}(E_f, E_m, \nu_f, \nu_m, V_f, V_m, C)\), where \(E_f, E_m\) are the elastic moduli of the fibre and matrix phases; \(\nu_f, \nu_m\) are the corresponding Poisson’s
ratios, $V_f, V_m$ the corresponding volume fractions and $C$ is a contiguity factor. The explicit expressions for $c_{ij}$ are given by Hashin and Rosen [1] and will not be repeated here. For the analysis of the bridged crack problem we select solutions of eq. (1) that are appropriate for the halfspace region $z \geq 0$ (i.e., the stresses $\sigma_{ij}$ and the displacements $u_i$ derived from $\varphi_i(r,z)$ should reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$). The appropriate solutions of eq. (1) are

$$\varphi_i(r,z) = \frac{1}{a_i^2} \int_0^{\infty} \xi A_i(\xi) e^{-\frac{\lambda_i z}{a_i}} J_0(\xi r/a_i) d\xi ; \quad (i=1,2)$$

where $A_i(\xi)$ are arbitrary functions and $\lambda_i = \xi / a_i \sqrt{\nu_i}$. The expressions for $u_z(r,z), \sigma_{zz}(r,z)$ and $\sigma_{rz}(r,z)$ take the forms

$$u_z(r,z) = \frac{\partial}{\partial z} \left\{ k_1 \varphi_1 + k_2 \varphi_2 \right\}$$

$$\sigma_{zz}(r,z) = \left\{ k_1 c_{33} - \nu_1 c_{13} \right\} \frac{\partial^2 \varphi_1}{\partial z^2} + \left\{ k_2 c_{33} - \nu_2 c_{13} \right\} \frac{\partial^2 \varphi_2}{\partial z^2}$$

$$\sigma_{rz}(r,z) = c_{44} \left\{ (1 + k_1) \frac{\partial^2 \varphi_1}{\partial r \partial z} + (1 + k_2) \frac{\partial^2 \varphi_2}{\partial r \partial z} \right\}$$

where

$$k_i = \frac{c_{11} V_i - c_{44}}{c_{13} + c_{44}} .$$

### 3 Body force loading of the external circular crack

Prior to examining the bridged external circular crack problem we should record here certain salient results related to the axisymmetric loading of the intact fibre reinforced solid by symmetrically placed body forces. Forces of magnitude $P$ act at the locations $z = \pm c$ on the $z$-axis. These are directed in the positive and negative $z$-directions, respectively, such that a state of tension exists in the plane of symmetry $z=0$. The state of stress $\sigma_{ij}$ due to this doublet of forces is such that $\sigma_{rz}(r,0) = 0$, and

$$\sigma_{zz}(r,0) = -p_0 \left[ \frac{\zeta_1 a^2 c_1}{(r^2 + c_1^2)^{3/2}} - \frac{\zeta_2 a^2 c_2}{(r^2 + c_2^2)^{3/2}} \right] = -p^*(r)$$

where
We now examine the problem of the external circular crack in which fibre continuity imposes an elastic bridging action in the region \( a \leq r \leq \infty \). The bridged external circular crack is subjected to symmetrically placed body forces of magnitude \( P \), which act at the location \((0, \pm c)\) as shown in Figure 1.

![Diagram of the bridged external circular crack](image)

Figure 1. The bridged external circular crack.

Since the problem exhibits a state of symmetry in \( \sigma_{zz} \) and \( u_z \) about \( z=0 \), and since \( a/l < 1 \), we can restrict the analysis to a single halfspace region in which the plane \( z=0 \) is subjected to the mixed boundary conditions

\[
\begin{align*}
  u_z(r,0) = 0 & ; \ 0 \leq r \leq a \\
  \sigma_{zz}(r,0) = p^*(r) + \frac{V_f E_f}{l} u_z(r,0) & ; \ a < r < \infty \\
  \sigma_{rz}(r,0) = 0 & ; \ r \geq 0.
\end{align*}
\]  

(11) \hspace{1cm} (12) \hspace{1cm} (13)

By making use of the solutions equ. (3) and the results equs. (4)-(5), it can be shown that the mixed boundary conditions, equs. (11)-(12) are equivalent to the system of dual integral equations
\[ \int_0^\infty \xi B(\xi) F(\xi) J_0(\bar{\xi} r / a) d\xi = \frac{p^*(r)}{2\mu^*}; \quad a < r < \infty \]  
(14)

\[ \int_0^\infty B(\xi) J_0(\bar{\xi} r / a) d\xi = 0; \quad 0 \leq r \leq a \]  
(15)

where

\[ A_1(\xi) = -\frac{v_1}{v_2} \frac{(1 + k_2) B(\xi)}{(1 + k_1) \xi^2}; \quad A_2(\xi) = \frac{B(\xi)}{\xi^2} \]

\[ F(\xi) = (1 - \frac{\psi}{\xi}); \quad \psi = \frac{a E_f V_f}{E_m} \sqrt{\frac{v_1 v_2 (k_1 - k_2)}{\Omega^*}} \]

\[ \mu^* = \frac{c_{44} \Omega}{2a^4 v_2 \sqrt{v_1 (1 + k_1)}} \]

\[ \Omega = \sqrt{v_1 (1 + k_1)} \left\{ \frac{k_2 c_{33} - v_2 c_{13}}{c_{44}} \right\} - \sqrt{v_2 (1 + k_2)} \left\{ \frac{k_1 c_{33} - v_1 c_{13}}{c_{44}} \right\} \]

\[ \Omega^* = \frac{\Omega c_{44}}{E_m}. \]

By introducing a transformation of the type

\[ B(\xi) = \frac{p_0}{\pi \mu^*} \int_0^1 \Phi^*(t) \cos(\xi t) dt \]  
(17)

the dual system equations (14)-(15) can be reduced to a single Fredholm integral equation of the second kind, i.e.,

\[ \Phi^*(t) - \frac{\psi}{\pi} \int_0^\infty K(t, \tau) \Phi^*(\tau) d\tau = g(t) \]  
(18)

where the kernel function \( K(t, \tau) \) is given by

\[ K(t, \tau) = 2 \int_0^\infty \frac{\cos(\xi t) \cos(\xi \tau) d\xi}{\bar{\xi}} \]  
(19)

and

\[ g(t) = \frac{\xi_1 \sqrt{v_1 \eta}}{(v_1 t^2 + \eta^2)} - \frac{\xi_2 \sqrt{v_2 \eta}}{(v_2 t^2 + \eta^2)} \]  
(20)

and \( \eta = c / a \).
The complete mathematical analysis of the bridged external circular crack is now reduced to the solution of the Fredholm integral equation (18). In the ensuing discussions, however, we shall restrict the attention to an examination of the influence of bridging action on the stress intensity factor at the crack boundary.

4 The stress intensity factor

The stress intensity factor for the opening mode of the bridged external circular crack is defined by

\[ K_I = \lim_{r \to a} (2(a - r))^{1/2} \sigma_{zz}(r, 0). \] (21)

By employing the results derived in the previous sections, it can be shown that

\[ [K_I]_{bridged\\ flaw}^{r, iso} = \frac{P}{2\pi^2 a^{3/2}} \left\{ 4\Phi^*(1) \right\}. \] (22)

In the limiting case when the elasticity of the bridged fibres \((E_f)\) reduces to zero, the parameter \(\psi = 0\) and the expression (equ. 22) yields the stress intensity factor for the dipole body force loading of an external circular flaw in a transversely isotropic elastic medium; i.e.,

\[ [K_I]_{unbridged\\ flaw}^{r, iso} = \frac{P}{2\pi^2 a^{3/2}} \left\{ \frac{2(c_{13} + c_{44})\nu_1\nu_2}{c_{33}c_{44}(\nu_1\nu_2)} \cdot \left( \frac{(k_1c_{33} - \nu_1c_{13})\eta}{\sqrt{\nu_1(\nu_1 + \eta^2)}} - \frac{(k_2c_{33} - \nu_2c_{13})\eta}{\sqrt{\nu_2(\nu_2 + \eta^2)}} \right) \right\}. \] (23)

In the limit of material isotropy, \(\nu_1, \nu_2 \to 1\) and \(c_{11} = c_{33} = \lambda + 2\mu\), where \(\lambda, \mu\) are the classical Lame constants. For the limiting case of an isotropic material with no bridging action

\[ [K_I]_{unbridged\\ flaw}^{iso} = \frac{P}{2\pi^2 a^{3/2}} \left[ \frac{(1 - 2\nu) + (3 - 2\nu)\eta^2}{(1 - \nu)(1 + \eta^2)^2} \right]. \] (24)

where \(\nu\) is Poisson’s ratio for the isotropic material.

Finally, in the limiting case where the fibre reinforced composite is composed of inextensible fibres, \(\psi \to \infty\), and the integral equation yields a trivial result \(\Phi^*(t) = 0\). Consequently,

\[ [K_I]_{bridged\\ flaw}^{next} \to 0. \] (25)
5 Numerical results

The stress intensity factor for the elastically bridged external circular flaw can be evaluated by performing a numerical solution of the integral equation (18). The various numerical procedures that can be adopted for the solution of the Fredholm integral equations of the second kind are well documented by Atkinson [18] and Delves and Mohamed [19]. In the present paper we employ a discretization method that involves the application of a Gaussian quadrature technique for the numerical evaluation of the resulting integrals. The elastic constants $c_{ij}$ of the transversely isotropic idealization are obtained by using the composite cylinder assemblage analogy given by Hashin and Rosen [1].

For purposes of illustration, the elastic properties for the matrix are assigned the following values: $E_m = 30 G N / m^2$; $\nu_m = 0.35$. The modular ratio $M = E_f / E_m$ is assigned values in the range $10^2$ to $10^4$; $V_f = 1/3, 2/3$ and $\nu_f = 0.20$. Other variables in the problem are the location of the body forces $\eta = c / a$ and the flaw aspect ratio $a/l$. The Figures 2 and 3 illustrate the manner in which the various geometric and material parameters influence the stress intensity factor at the boundary of the bridged crack. These results exhibit trends that have been identified previously in connection with the limiting conditions.

Figure 3. The stress intensity factor for the bridged crack
6 Conclusions

This paper examines, via a theoretical model, the possible influence of fibre bridging on the stress intensity factor for an external circular crack loaded by a doublet of forces. This model serves as a useful first approximation for the stress analysis of a unidirectional fibre reinforced composite region in which the matrix has fractured to create the external circular crack with elastic bridging. Numerical calculations indicate that the stress intensity factor is considerably reduced as a consequence of the fibre bridging action. The influence of this bridging action on criteria for extension of such bridged defects, which is of interest to the design of fibre reinforced composites, merits further investigation.

References