A feeding impedance model for electrified rail track

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Abstract

A model for investigating resonance conditions in electrified railway track is presented. The short-circuit and open-circuit input impedances as functions of frequency and feeding track length are calculated from frequency-dependent track phase impedances and admittances found from verified finite element modelling. The corresponding characteristic impedances and propagation constants are also calculated. The results of the study are useful to practising engineers in assessing power and data transmission properties of rail track.

1 Introduction

Knowledge of the input impedance of electric traction networks is of interest to electrical and communication engineers engaged in the design of traction power and signalling systems. Railway track may be regarded as a coupled transmission line. The fundamental data of such a model are the phase self and mutual impedances and admittances of the rails, catenary and other parallel conductors. These data are difficult to obtain because of the inaccessible remote earth reference. The calculations presented in this paper rely on track data obtained from finite element models to assess the input impedance of various configurations of traction power and signalling feeding circuits.

The classic method of estimating traction system input impedance is to make simplifying approximations about the line model. The feeding impedance for traction power flow is usually evaluated in terms of the equivalent self impedances of the multiconductor lines [1-3]. Detailed studies require a full track model, the data for which may be obtained either by analytical modelling implemented by computer software [4], or more flexibly from finite-element modelling [5]. In track signalling circuits, balanced conditions, considering only the loop impedance and admittance, are usually assumed [6, 7]. These have limited use, however, and to make detailed system studies of electromagnetic compatibility between power supply, traction drives, signalling and communications circuits, knowledge of the
complete set of self and mutual track impedances and admittances is necessary [8].

In this paper, the frequency-dependent track phase impedances and admittances are presented for single unelectrified and overhead catenary electrified tracks. The data has been obtained by finite-element modelling verified analytically and by practical testing. Simple algebraic expressions are then derived for various feeding impedances and admittances applicable to both power and track signalling circuits. The signal propagation characteristics are conveniently expressed in terms of characteristic impedance and propagation constant, and these quantities are evaluated for the feeding conditions considered.

2 Transmission line model of rail track

2.1 Transmission line equations

Rail track and traction line can be represented as a multiconductor transmission line (MTL) with frequency-dependent self and mutual impedances and admittances expressed per unit track length. The equations for voltage and current drop along the line are

\[
\frac{dV_i}{dx} = - \left( Z_{ii} I_i + \sum_{j=1}^{n} Z_{ij} I_j \right) \quad j \neq i \tag{1}
\]

and

\[
\frac{dI_i}{dx} = - \left( Y_{ii} V_i + \sum_{j=1}^{n} Y_{ij} V_j \right) \quad j \neq i \tag{2}
\]

where \( V_i, V_j \) are the phase voltages and \( I_i, I_j \) are the phase currents along lines \( i, j \); \( Y_{ii}, Y_{ij} \) are the self admittance of line \( i \) and the mutual admittance between lines \( i \) and \( j \) per unit track length (defined by the admittance matrix); \( Z_{ii}, Z_{ij} \) are the self impedance of line \( i \) and the mutual impedance between lines \( i \) and \( j \) per unit track length; and \( x \) is the distance along the track. In vector form, Equations (1) and (2) become

\[
\frac{d [V]}{dx} = - [Z][I] \tag{3}
\]

and

\[
\frac{d [I]}{dx} = - [Y][V] \tag{4}
\]

where \([V]\) and \([I]\) are the vectors of phase voltages and currents, and \([Z]\) and \([Y]\) are the matrices of phase impedances and admittances per unit track length. Combining Equations (3) and (4), the wave equations are obtained:

\[
\frac{d^2[V]}{dx^2} = [Z][Y][V] = [P][V] \tag{5}
\]

and
where \([P]\) is a propagation matrix and \([P]^T\) is its transpose. Since \(Z_{ij} = Z_{ji}\) and \(Y_{ij} = Y_{ji}\) (for \(i \neq j\)), the matrices \([Z]\) and \([Y]\) are symmetric. However, in general, \([P]^T \neq [P]\).

### 2.2 Phase impedances and admittances

The phase self impedance forming the diagonal elements of the impedance matrix includes resistance and inductance. The off-diagonal terms represent mutual impedances from coupling between pairs of independent rail circuits, each with its own earth return path. The values of the track impedance and admittance matrix elements are determined by the material properties (conductivity, permeability and permittivity) of the rails and earth and the physical size and relative position of the rails and other conductors. The matrices contain non-zero elements at all positions because of the coupling between the individual rails and other conductors due to the mutual components. The equivalent values of the self and mutual impedance and admittance for overhead catenary electrified track have previously been determined by finite-element modelling, verified experimentally for a short length test track [9, 10]. The results are given in Figure 1.

![Figure 1. Phase impedance and admittance for catenary electrified single track](image-url)
2.3 Track impedance

The self and mutual inductances decrease and resistances increase with increasing frequency and ground conductivity due to the skin effect. The self inductance is the sum of contributions from internal and external components, the latter arising from the effects of the return current path through the earth. The internal component is dominant at low frequencies - for example in rail it is about 30% of the total at 30 Hz. The ground skin effect determines both external self and mutual inductances, and arises from induced ground currents near the surface. Physically, the mutual inductance between two rails is a measure of the flux linking one circuit comprising the rail with its earth return from currents in another rail. The mutual resistance arises from losses in the ground from induced eddy currents and is zero at DC.

2.4 Track admittance

In electrified track, the conductance between the catenary and the running rails is negligible, due to the supporting insulators, whereas between the running rails, the conductance will normally exceed the susceptance. The running rail self and mutual capacitance reduce with increasing frequency, reflecting the influence of the ground permittivity which varies with frequency due to the water content of the ground geological strata. The capacitance is also weakly dependent on the ground conductivity. The catenary-to-rail mutual capacitance, however, is not influenced greatly by the ground property values and stays approximately constant. The self and mutual conductances increase slightly with increasing frequency due to the ground conductivity variation. They also increase in magnitude with increasing ground conductivity.

3 Characteristic impedances, propagation constants and line input impedance

3.1 Characteristic impedance and propagation constant

The characteristic impedance of a line with impedance per unit length of $Z$ and admittance per unit length of $Y$ is defined by

$$ Z_0 = \sqrt{\frac{Z}{Y}} $$

(7)

For a single phase line, the characteristic impedance, when placed at the line termination, enables the line to behave as infinite, with no reflected waves. The phase characteristic impedances for a multiconductor line are the network which when placed at the line termination have the effect of preventing reflections on any of the phases. It is a full matrix with elements corresponding to both phase and line impedances. For a particular excitation, the characteristic impedance matrix may be found from the phase impedance and admittance matrices. In the general case, signals will appear on unexcited phases arising from wave reflections due to signals on other excited phases.

The propagation constant of a line with impedance per unit length of $Z$ and admittance per unit length of $Y$ is defined by

$$ \gamma = \alpha + j\beta = \sqrt{ZY} $$

(8)
where $\alpha$ is the attenuation and $\beta$ is the phase constant. In the general case of a multiconductor line, the propagation constant is a matrix.

### 3.2 Input impedance

The input impedance of a transmission line can be found from the characteristic impedance and propagation constant as:

$$Z_{\text{in}} = Z_0 \frac{Z_L + Z_0 \tanh(\gamma L)}{Z_0 + Z_L \tanh(\gamma L)}$$  \hspace{1cm} (9)

where $Z_0$ is the characteristic impedance, $\gamma$ is the propagation constant, $Z_L$ is the termination impedance and $L$ is the line length.

### 4 Short-circuit and open-circuit feeding impedances and admittances

The 3-wire line model of Figure 2 has been used to calculate the feeding impedance for various configurations starting from the track phase impedances and admittances. Three cases have been considered corresponding to track signalling (running rails only), catenary power feeding and third-rail power feeding, the latter two both with single and double rail return. The results presented in this paper focus on power feeding circuits in a catenary system.

![MTL system, signalling and power feeding circuit models](image)

*Figure 2: MTL system, signalling and power feeding circuit models*
For three-wire track excitation, the voltages along the line are expressed from Equation (3) as

\[
\frac{d}{dx} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}
\]

(10)

and the currents from Equation (4) as

\[
\frac{d}{dx} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} & Y_{13} \\ Y_{21} & Y_{22} & Y_{23} \\ Y_{31} & Y_{32} & Y_{33} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}
\]

(11)

where the suffix 1 refers to the catenary and 2 & 3 to the running rails.

### 4.1 Single rail return

Traction power fed along the catenary may return along either one running rail alone or both rails. For single-rail return conditions, the feeding impedance is equal to the loop impedance:

\[
Z_{fs} = Z_{11} + Z_{22} - 2Z_{12}
\]

(12)

where suffix 2 denotes the rail carrying return current. Note that for identical running rails, \(Z_{22} = Z_{33}\) and for catenary feeding, the mutual impedances between catenary and each rail are equal, \(Z_{12} = Z_{13}\).

The feeding admittance must take into account mutual coupling to the unused rail. It may be shown that it is

\[
Y_{fs} = Y_{M12} + Y_p + \frac{\left(Y_Q + Y_{1e}\right)\left(Y_R + Y_{2e}\right)}{Y_{1e} + Y_{2e} + Y_Q + Y_R}
\]

(13)

where

\[
Y_p = \frac{Y_{M13}Y_{M23}}{Y_{M13} + Y_{M23} + Y_{3e}}
\]

(14)

\[
Y_Q = \frac{Y_{M13}Y_{3e}}{Y_{M13} + Y_{M23} + Y_{3e}}
\]

(15)

and

\[
Y_R = \frac{Y_{3e}Y_{M23}}{Y_{M13} + Y_{M23} + Y_{3e}}
\]

(16)

where \(Y_{1e}\) and \(Y_{Mij}\) are the equivalent self admittance component of line i and the equivalent mutual admittance between lines i and j per unit track length, derived from the admittance matrix elements using
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\[ Y_{ij} = -Y_{Mij} \quad (17) \]

and

\[ Y_{ie} = Y_{ii} + \sum_{j=1}^{n} Y_{ij} \quad j \neq i \quad (18) \]

Equation (13) may be simplified for identical running rails and catenary track since \( Y_{2e} = Y_{3e} \) and \( Y_{Ml2} = Y_{Ml3} \).

4.2 Double rail return

The feeding impedance for double-rail return may be obtained by substituting \( V = (V_1 - V_2) \) and \( I_1 = -(I_2 + I_3) \) in Equations (10) and (11). It may be shown [11] that for the general case

\[ Z_{Fd} = -kZ_{11} - Z_{22} + (1 + k)Z_{13} - (1 - k)Z_{12} + (1 - k)Z_{23} \quad (19) \]

where

\[ k = \frac{Z_{22} - Z_{23} + Z_{13} - Z_{12}}{Z_{11} + Z_{22} - 2Z_{12}} \quad (20) \]

The feeding impedance may be found by substituting the conditions for identical running rails and catenary track, \( Z_{22} = Z_{33} \) and \( Z_{12} = Z_{13} \). The feeding admittance is

\[ Y_{Fd} = 2Y_{Ml2} + \frac{2Y_{le}Y_{2e}}{Y_{le} + 2Y_{2e}} \quad (21) \]

5  Feeding impedance, characteristic impedance and propagation constant results

Figure 3 shows the short circuit and open circuit feeding impedances of catenary electrified track with single rail return as evaluated from Equations (12) and (13), with track length (to 5 km) and frequency (to 26 kHz) as parameters, and using the track phase data of Figure 1. The expected transmission line wave effects are clearly shown, with the magnitude of the peaks decreasing with frequency due to increased attenuation.

Figure 4 shows the transmission line resonant effect more clearly. The short-circuit impedances are zero at DC and increase to maxima at the quarter wavelengths, with the reverse behaviour for the open-circuit impedances. The longest length track exhibits the shortest wavelength. The quarter wavelength peaks decrease in magnitude with higher track lengths due to increased resistive and conductive losses. The detail for the 5 km line shows convergence of the quarter wavelength points with the magnitude of \( Z_0 \).

Figure 5 shows the calculated catenary feeding circuit transmission line constants, for the cases of single-rail and double-rail return, as functions of frequency. The characteristic impedance is almost constant, although its angle increases to about 100 Hz, then decreases. The attenuation and phase constants both increase with frequency.
### Table 1: Short-circuit and open-circuit input impedances of catenary electrified track as function of frequency and track length

<table>
<thead>
<tr>
<th>Feeding length (m)</th>
<th>Frequency (Hz)</th>
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#### Figure 3: Short-circuit and open-circuit input impedances of catenary electrified track as function of frequency and track length

#### Figure 4: Short-circuit and open-circuit input impedance for catenary electrified track as function of frequency

* A, Curves for 1, 2, 3, 4 and 5 km track lengths
* B, 5 km track length (with characteristic impedance superimposed)
6 Discussion and conclusions

The results are of interest to traction engineers concerned with power feeding since line resonance conditions may give rise to overvoltages if excited by harmonics produced by traction vehicle converters at the critical resonant frequencies. Figure 3 shows that the frequency and magnitude of the line resonances are dependent on feeding length. In a practical case, however, conditions are more complex, since the line length may not necessarily change with traction vehicle position due to bidirectional power feeding and/or the presence of an infinite line on one side of the vehicle.

For each track feeding length, the values of the short circuit and open circuit impedances coincide at critical frequencies corresponding to quarter wavelengths of the standing wave pattern (Figure 4). For catenary electrified track, these resonances are determined from the geometric configuration; as shown in Figure 1, the dominant components are the catenary-to-rail capacitance ($C_{m23}$) and the catenary and rail self inductances ($L_{33}$ and $L_{11}$).

Since the track phase data can vary by up to four orders of magnitude in the frequency range of interest, the use of frequency-dependent track data is mandatory for this study. The results were obtained using data obtained from finite-element analysis and verified experimentally and by alternative analytic modelling. Current work is oriented to refining these track models and proving their accuracy under practical engineering conditions.
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References