Numerical model of tram-track interaction with failures on rail surface

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Abstract

The paper deals with numerical model of a tram moving on the track. The rail is modelled as an elastically supported continuous beam with lumped masses. A vehicle is approximated as a discrete system consisting of three rigid masses connected with linear elastic springs. Dissipation mechanisms in the system are modelled with the linear dashpots. The input parameters are obtained from the manufacturer and from measurements in the laboratory and in situ. Two basic assumptions that are used are: a) the Euler-Bernoulli beam theory and b) the separation of the wheel from the rail is not permitted. The system of differential equations of motion is solved by an implicit version of step by step method based on the direct integration technique. The algorithm is conceived on the first order predictor – (re) corrector method. The contact condition is modelled by the force method. Results of the calculations are compared with the results of measurement on the tram track. Numerical predictions for displacements and accelerations are in good agreement with the experimental findings.

Keywords: tram-track interaction, rail, weld, numerical model, contact force.

1 Introduction

Despite the fact that, compared to the classical railway track, the tram track features the low speed of vehicle and low load per wheel, the Zagreb tram track is characterized by an extreme traffic load. Namely, several tram tracks in the centre of Zagreb have a traffic volume of up to 15 million gross tons a year per cross section with the frequency of tram vehicle running under 1 minute [1]. An inspection of 64-kilometre tram track reconstruction carried out from 1994 to 2002 resulted in the findings of massive damages caused by the vehicle motion over failures on the track surface. Not only is the damage sanation expensive and
requires the halting of the tram traffic, but it also interferes with road traffic on the sections shared by both vehicle types. The highest percentage of both the track and the track elements closure damages take place in the welded track sections. Due to the unfavourable ratio of dynamic and static loads within track structures there is a high dynamic factor on this places. The study aims to evaluate additional dynamic effects on the track that result from the recorded geometric irregularities on the rail surface.

2 Numerical model

It is known that the results of numerical calculations can significantly facilitate the selection of the track structure. To determine the dynamic effects of the vehicle on the tram track a numerical model of the vehicle motion along the track was defined. The model consists of the connected track-vehicle models, fig. 1.

\[
\begin{align*}
0 & = m_3 \ddot{z}_3 + c_2 (\dot{z}_3 - \dot{z}_2) + k_2 (z_3 - z_2) \\
0 & = m_2 \ddot{z}_2 + c_1 (\dot{z}_2 - \dot{z}_1) - c_2 (\dot{z}_3 - \dot{z}_2) - k_2 (z_3 - z_2) + k_1 (z_2 - z_1) \\
0 & = m_i \ddot{z}_i - c_1 (\dot{z}_2 - \dot{z}_1) - k_i (z_2 - z_1) = n
\end{align*}
\]

or in a matrix form:

Figure 1: Train-track model.

2.1 Vehicle model

The study considers a plane model of a vehicle with three vertical degrees of freedom \((z_1, z_2, z_3)\). The vehicle is modelled by translational masses of wheel set \((m_1)\), bogie \((m_2)\) and tram body \((m_3)\) linked with elastically dissipative bonds of determined stiffnesses \((k_1\) and \(k_2)\) and viscosities \((c_1\) and \(c_2)\). Due to dynamic interactions of the vehicle and track the wheel is loaded also with the contact force \(n(x, t)\).

The mathematical formulation of this model is based on three differential equations of motion (space and time dependencies are omitted):

\[
\begin{align*}
m_3 \ddot{z}_3 + c_2 (\dot{z}_3 - \dot{z}_2) + k_2 (z_3 - z_2) &= 0 \\
m_2 \ddot{z}_2 + c_1 (\dot{z}_2 - \dot{z}_1) - c_2 (\dot{z}_3 - \dot{z}_2) - k_2 (z_3 - z_2) + k_1 (z_2 - z_1) &= 0 \\
m_i \ddot{z}_i - c_1 (\dot{z}_2 - \dot{z}_1) - k_i (z_2 - z_1) &= n
\end{align*}
\]
\[ \mathbf{M}_v \ddot{\mathbf{z}}(x,t) + \mathbf{C}_v \dot{\mathbf{z}}(x,t) + \mathbf{K}_v \mathbf{z}(x,t) = \mathbf{f}_v - \mathbf{n}(x,t), \]  

where \( \mathbf{M}_v \), \( \mathbf{C}_v \) and \( \mathbf{K}_v \) are constant mass, damping and stiffness matrices, and \( \ddot{\mathbf{z}}(x,t) \), \( \dot{\mathbf{z}}(x,t) \) and \( \mathbf{z}(x,t) \) are the vectors of accelerations, speeds and displacements of the vehicle model. For the given model the mentioned matrices have the following form:

\[
\mathbf{M}_v = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_j \end{bmatrix}; \quad \mathbf{C}_v = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_1 + c_2 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix}; \quad \mathbf{K}_v = \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_1 + k_2 & -k_1 \\ 0 & -k_1 & k_1 \end{bmatrix},
\]

and vectors (with omission of \( x \) and \( t \)) are:

\[
\dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_3 \\ \dot{z}_2 \\ \dot{z}_1 \end{bmatrix}; \quad \ddot{\mathbf{z}} = \begin{bmatrix} \ddot{z}_3 \\ \ddot{z}_2 \\ \ddot{z}_1 \end{bmatrix}; \quad \mathbf{z} = \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}; \quad \mathbf{f}_v = g \begin{bmatrix} m_3 \\ m_2 \\ m_1 \end{bmatrix}; \quad \mathbf{n} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

For the given model the mentioned matrices have the following form:

\[
\mathbf{M}_v = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_j \end{bmatrix}; \quad \mathbf{C}_v = \begin{bmatrix} c_2 & -c_2 & 0 \\ -c_2 & c_1 + c_2 & -c_1 \\ 0 & -c_1 & c_1 \end{bmatrix}; \quad \mathbf{K}_v = \begin{bmatrix} k_2 & -k_2 & 0 \\ -k_2 & k_1 + k_2 & -k_1 \\ 0 & -k_1 & k_1 \end{bmatrix},
\]

and vectors (with omission of \( x \) and \( t \)) are:

\[
\dot{\mathbf{z}} = \begin{bmatrix} \dot{z}_3 \\ \dot{z}_2 \\ \dot{z}_1 \end{bmatrix}; \quad \ddot{\mathbf{z}} = \begin{bmatrix} \ddot{z}_3 \\ \ddot{z}_2 \\ \ddot{z}_1 \end{bmatrix}; \quad \mathbf{z} = \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}; \quad \mathbf{f}_v = g \begin{bmatrix} m_3 \\ m_2 \\ m_1 \end{bmatrix}; \quad \mathbf{n} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

**Figure 2:** Vehicle model: a) discretisation; b) dynamic forces.

The notations \( \mathbf{f}_v \) and \( \mathbf{n}(x,t) \) represent the vehicle weight and dynamic component of the contact force so that the following ensues:

\[
\mathbf{f}_v = k \mathbf{f}_v \quad \text{and} \quad \mathbf{n}(x,t) = k \mathbf{n}(x,t),
\]

where \( k \) is a unit vector in the direction of the \( z \) axes, and \( \mathbf{f}_v \) and \( \mathbf{n}(x,t) \) are the magnitudes of the mentioned forces. The static component of the contact force \( \mathbf{f}_v \) is equal to the vehicle weight and is given by:
Described discretisation of the vehicle is shown in figure 2.

2.2 Track model

As the study considers the Zagreb tram track (the tram track on the concrete slab), the assumption of an absolutely rigid reinforced concrete slab was introduced. The rail is modeled as elastically supported continuous beam with lumped masses. For the homogenous rail model ($E=\text{const.}$) of the constant transversal cross-section ($I=\text{const.}$), dynamically loaded with $f(x,t)$ and with the vehicle weight $f_v$, the following differential equation of a dynamic equilibrium can be written:

$$El\dddot{u}(x,t) + m\dddot{u}(x,t) = f(x,t) + f_v$$

(7)

where $u^{IV}(x,t)$ is the fourth spatial derivative of the deflection, and $\dddot{u}(x,t)$ the second time derivative (acceleration). Dynamic force can be determined as:

$$f(x,t) = \begin{cases} 
    c_L\dot{u}(x,t) + k_Lu(x,t) + n(x,t), & u(x,t) > 0, \\
    c_t\dot{u}(x,t) + k_Ku(x,t) + n(x,t), & u(x,t) < 0, \\
    c_t\dot{u}(x,t) + n(x,t), & u(x,t) = 0, \\
    n(x,t), & \text{otherwise},
\end{cases}$$

(8)

where $k_L$ and $k_K$ are the constant values of the support stiffnesses in compression and tension (figure 3a), and $c_L$ is the viscosity of the support (figure 3b). As seen in the equation (8) and figure 4, the activation of each separate connection $k_L$ or $k_K$ depends on the rail position. If there is a deflection of the rail due to the load, the support stiffness activates in the compression ($k_L$), whereas the stiffness in the tension $k_K = 0$. With the lifting of the rail the layer stiffness activates in the tension ($k_K$), and the stiffness in compression $k_L = 0$. The magnitude $x_i$ indicates the position of the support.

Figure 3: Stiffness and viscosity of the layer position.

Discretisation of the rail segment between two supports is shown in figure 5. The rail mass is concentrated in the nodes. The supports are modeled by Kelvin's viscoelastic massless body (fig. 4 and 5). Figure 6a shows how the forces act on the node within the span, while fig. 6b displays the way the forces act on the support.
For the given system the rail mass matrix $\mathbf{M}_t$ has a known diagonal form:

$$\mathbf{M}_t = \mathbf{I} \mathbf{m}_t,$$

where $\mathbf{I}$ is a unit matrix, and $\mathbf{m}_t = \{m_{t,1}, \ldots, m_{t,n}\}$.

The initial coordinates of the rail model are determined by the measurement results of the recorded rail running surface on the positions of the welds. The profilograph (brand name CEMAFER) recorded the total of 54 welds. The photograph of a weld on the Zagreb rail track and the recorded running surface are seen in figure 7.

Besides the failures on the running surface, the cases of possible rail crackings were considered. A failure in the shape of the groove with the width of $\lambda=30\text{mm}$ was created on the running surface, fig. 8. From the known width of the groove and the radius of the tram wheel ($R=350\text{mm}$) we got the trajectory of
the wheel motion shown in figure 8. That led to the size of the maximal vertical
wheel motion $y_1=0.322\text{mm}$ and the angle $\alpha$ at which the vehicle wheel meets the
irregularities. The geometric irregularities leading to vibration of vehicle-rail
system are thus taken into account.

Figure 7: Picture of weld and vertical weld geometry, “weld_49”.

Figure 8: Support of rail trajectory of wheel.

2.3 Vehicle - rail interaction

Following definitions of track and vehicle mathematical models, it is necessary
to integrate them into a complete motion model. The connection is based on the
assumption of the full compatibility of the displacement in the wheel-rail contact
point. The given condition can be expressed as:

$$u(x,t) = z_1(x,t), \quad (10)$$

where $u(x,t)$ is a rail displacement, and $z_1(x,t)$ is a wheel displacement in the
position $x$ at the moment $t$. That means that the rail and the wheel have a
common displacement, i.e., the separation of wheel and the rail is not allowed.
The equation system (2), (7) and (10) represents the mathematical model of the
vehicle motion along the track.

To achieve the complete solution, the initial and the boundary conditions
should be known. The initial conditions for the track are defined by:

$$\dot{u}(x,0) = 0, \quad u(x,0) = 0, \quad (11)$$

while the boundary conditions for the sufficient length $L$ (fig. 4) of the rail can
be expressed as:
For the vehicle in the static equilibrium starting from the condition of rest, the initial conditions are:
\begin{equation}
\dot{z}(0,0) = 0, \quad z(0,0) = K^{-1} f_v.
\end{equation}
\begin{equation}
\end{equation}

The vehicle speed in the horizontal direction is considered to be constant:
\begin{equation}
v_x(x,t) = v = \text{const}.
\end{equation}
\begin{equation}
\end{equation}

The equation (13) represents a static case of the equation (2), and this is at the same time the boundary condition for displacements. As for the boundary condition for the vehicle wrt. forces, it is given by the static component of the contact force according to the expression (6):
\begin{equation}
p(0,0) = f_v.
\end{equation}
\begin{equation}
\end{equation}

2.4 Numerical model of vehicle track motion system

Due to the non-linearity of the model there is no principle of superposition. Numerical model should be solved by some step-by-step methods. This paper uses the procedure of direct predictor - (re) corrector integration, and the procedure of direct integration is applied. Predictor – (re) corrector is sometimes called predictor, multiple corrector method. The contact condition (10) modeled by the force method is used to determine contact force \( n(x,t) \). Differential equation (7) can be written in the following form:
\begin{equation}
\ddot{u}(x,t) = \frac{1}{m_t} \left[ f(x,t) + f_{gy} - EIu^{IV}(x,t) \right].
\end{equation}
\begin{equation}
\end{equation}

Applying the method of finite differences, the equation (16) for the given node is discretized in the following form:
\begin{equation}
\ddot{u}_i(t) = \frac{1}{m_{t,i}} \left[ f_i(t) + f_{gy} - \frac{EI}{\Delta x^4} \left[ (u_{i-2}(t) - 4u_{i-1}(t) + 6u_i(t) - 4u_{i+1}(t) + u_{i+2}(t)) \right] \right].
\end{equation}
\begin{equation}
\end{equation}

where \( \Delta x \) is the mesh step. Nodal indices are \( i = -1, ..., n + 1 \). The corresponding mass in the node \( i \) is \( m_{t,i} = \Delta x \cdot m_t \). Dynamic load is, according to (8):
\begin{equation}
f_i(t) = \begin{cases} 
  c_L \dot{u}_i(t) + k_L u_i(t) + n_i(t), & u_l(t) > 0, \\
  c_L \dot{u}_i(t) + k_k u_i(t) + n_i(t), & u_l(t) < 0, \\
  c_L \dot{u}_i(t) + n_i(t), & u_l(t) = 0, \\
  n_i(t), & \text{otherwise}, 
\end{cases}
\end{equation}
\begin{equation}
\end{equation}

where \( l \) is the index of the support node. How the forces act on the node \( i \) is shown in the figure 6a. Mathematical model of the vehicle (2) matches with its numerical model. The procedure of solving the mentioned system in a time step \( \Delta t \) can be presented in the following stages:
\begin{enumerate}
  \item separate solving of track-vehicle equations for the case when the contact force \( n_i(t) = 0 \);
  \item solving of the same equations for the case when the contact force has a unit value \( n_i(t) = 1 \);
\end{enumerate}
c) the real magnitude of contact force between wheel and rail should be determined using the conditions of compatibility (19).

Discussed procedure can be considered as the application of the force method on the system whose static indeterminacy is caused by an unknown contact force \( n(t) \) at the rail-wheel joint. The method presumes that we can linearize the problem at each time step, which is sensible to presume for the small steps; therefore the principle of superposition is valid [3]. So far, the spatial discretisation has been considered (i.e. variables still depend on \( t \)), while the time discretisation has been carried out by the predictor-corrector method, which will not be discussed in detail [4].

3 Verification of the numerical model

The comparison between the experiment and numerical simulation was carried out for the rail displacement and for the vertical acceleration of the rail. The rail deflections were measured with LVDT inductive sensors with sensitivity 700mV/1mm, while the rail accelerations were measured with accelerometer whose sensitivity is 10mV/1m/s^2 and with range 0.1kHz-20kHz. The measurements were performed using ADC-216 of the PICO Technology and software PicoScope company. The measurements were carried out on the new tram track. In this way the effects of worn out elements of track fasteners were eliminated as they were not included into the numerical model.

3.1 Rail displacements

The comparison of rail displacement was made for the vehicle speed of 36km/h and for a loading per wheel 46kN. The magnitudes of track displacement obtained by measurements and calculations are shown in figure 9 (smooth running surface) and figure 10 (irregularity on running surface). A displacement diagram obtained by measurements shows the obstruction of 50Hz frequency matching the supply frequency of measuring devices. The calculations led to the increase in the rail deflection of up to 45% in the cases of failures on the rail running surface. The comparison of the measurement and calculation results shows in measurement a recorded displacement due to the passing of each vehicle wheel, whereas a displacement under a one wheel was observed in calculations, as stated in the item 2.1.

![Figure 9: Rail displacements (smooth running surface).](image-url)
3.2 The acceleration of the rail

The comparison of the rail accelerations was made for the vehicle speed of 25km/h and 41kN load per wheel. Data obtained by measurements and calculation is shown in figure 11 (smooth running surface) and figure 12 (irregularity on running surface). It is clear that the magnitudes of the rail acceleration in the case of the geometric irregularities on the running surface increase up to 300%.

3.3 The analyses of measurement and calculation results

The comparison of the results of measurement and calculations shows that the results of the calculation were only 10% to 15% higher than the measured ones, both for the vehicle motion on the track with smooth running surface, according
to [3], and for the case of failures on the running surface discussed in the paper. These findings can be explained by the fact that there is no effect of the another wheel set in the calculations as the vehicle is simulated with the load over one wheel only. A further difference between the measurement and calculation result is that the measurement data were recorded in the time domain with $\Delta t = 0.001s$, while the time interval for the calculation was $2.0 \times 10^{-6}s$. The rail displacement and acceleration diagrams show clearly the moment of the vehicle wheel meeting the failure in the measurement and the calculation as well. The vehicle motion over a failure causes the separation of the wheel from the rail, with immediately following renewed contact with the rail accompanied by unavoidable impact. The consequence of the impact is a significant increase in displacement and rail acceleration. As the model presumes the equality of displacements in the contact point, the calculation model applies only until the impact moment, that means, until the moment of separation of the wheel from the rail.

4 Contact force

The definitions of dynamic and static components of the contact force led to the definition of the total contact force in the form:

$$ p(x,t) = n(x,t) + f_v. $$

(19)

The deflection due to the rail weight is very low and thus can be neglected. The study covers the investigation of the contact forces between the wheel and rail, but only for the case of compressive forces. In the case of rail-wheel separation, the contact force is equal to zero, and the compatibility condition is disrupted (10). The calculation of the contact force magnitude for the vehicle speed of 36km/h, the 6m rail length, and for the geometry of the running surface from figure 7, is shown in fig. 13. For 54 recorded running surface failures in the zones of the rail welds, and for the vehicle speed of 55km/h (maximal speed of tram KT-4) an average increase of the contact force of 138kN was obtained. For the respective tram vehicle it represents the contact force increase up to 175% with respect to the case when the vehicle moves along the smooth running surface. The obtained average value of dynamic factor is 2.9.

![Figure 13: Contact force diagram.](image)
5 Conclusion

The increase of the load on the track due to the vehicle motion over the failures on the running surface leads to the damages of the tram structure. A systematic study of running surface geometry and the constant renewal of inconvenient spots can remarkably contribute to the reduction of dynamic effects on the track. These elements have great impact on the reduction of the noise level, reduction of the maintenance costs and an increase of the design life of the construction and journey comfort as well. Further investigation involves the extension of the numerical model for the case when the wheel separates from the rail. Therefore the study will concentrate on the investigation of the trajectory of wheel motion from the impact moment onto the failure to the position of the regained rail contact. Further on, the research should result in finding the force magnitude at the second impact.

References