Vibrational characteristics of overhead-line equipment by Transfer Matrix Method

T. Morikawa

Power Supply Research Division
Railway Technical Research Institute, Japanese Railway

Abstract

Vibrational characteristics of overhead-line equipment cannot be solved analytically because the system constitutes neither a distributed constant system nor a concentrated constant one. Calculations have been carried out for this system, by means of transfer matrix method to clarify the vibrational response of simple catenary equipment by point excitation. In the analysis, both the contact and messenger wires are assumed to be tensioned strings with their bending stiffness neglected. Therefore, each section of dropper spacing has four eigenfunctions and coefficients, which are related to neighboring ones. By applying these relations of dropper spacing to the far away one step by step, the response of simple catenary equipment by point excitation is analytically gained. As a result, it is revealed that simple catenary equipment has two similar fundamental vibration modes. In one mode, the motive directions of contact wire of a span accord with those of others. In the other mode, the motive directions are opposite in every neighboring span. The former is defined as a ‘mode in the same phase’ and the latter as a ‘mode in the reverse phase’. Their frequency is about 1Hz under standard wire tensions. The fundamental mode in same phase is a little lower than that of reverse phase in frequency. The results of this paper are valid in the frequency range of 0Hz to about 30Hz, where, that is, a wire behaves as a string.
1 Introduction

Generally, vibrational modes of overhead line equipment are not adequately known, while its stiffness distribution along a line is well known. When pantographs pass through a span at the speed of 300km/h, what modes of it are excited or at how large amplitude they vibrate. We must clarify its vibrational modes prior to the estimation of mutual dynamics with a massive pantograph. Therefore, this paper describes the vibrational modes of simple catenary equipment to solve analytically by the transfer matrix method. This was first applied to the vibratory response of an infinitive length rail (1) and now to overhead line equipment.

2 Analytic calculations

2.1 Transfer matrix between neighboring dropper spacings

Each wire is treated as a string with its rigidity neglected. This assumption is valid in the low frequency range of 0.30Hz. Then, assumed that the dropper bar and dropper clamp have no weights. Under these conditions in arbitrary dropper spacing, their motions are expressed by the linear wave equations (1).

\[
\rho \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2}, \quad \rho \frac{\partial^2 z}{\partial t^2} = T_M \frac{\partial^2 z}{\partial x^2}
\]

Where

\(T\) : Tension of contact wire, \(T_M\) : Tension of messenger wire
\(\rho\) : Linear density of contact wire, \(\rho_M\) : Linear density of messenger wire
\(y\) : Displacement of contact wire, \(z\) : Displacement of messenger wire
\(x\) : Horizontal axes, \(t\) : Time

Figure 1 : Model of simple catenary

Figure 2 : A coordinate system in neighboring dropper spacing
When a sinusoidal motion as an angular velocity \( \omega \) is considered, their solutions are given as follows:

\[
y_n = (A_n \sin k_T x + B_n \cos k_T x) e^{-j\omega t} \tag{2}
\]

\[
z_n = (C_n \sin k_M x + D_n \cos k_M x) e^{-j\omega t}
\]

\[
y_{n+1} = (A_{n+1} \sin k_T x + B_{n+1} \cos k_T x) e^{-j\omega t} \tag{3}
\]

\[
z_{n+1} = (C_{n+1} \sin k_M x + D_{n+1} \cos k_M x) e^{-j\omega t}
\]

Where

\[
k_T = \frac{\omega}{c_T}, \quad k_M = \frac{\omega}{c_M}, \quad c_T = \sqrt{\frac{T_T}{\rho_T}}, \quad c_M = \sqrt{\frac{T_M}{\rho_M}}
\]

The following boundary conditions exist between the coefficients of dropper spacing on both sides at a dropper point.

\[
z_n \bigg|_{x = a} = y_n \bigg|_{x = a}, \quad z_n \bigg|_{x = a} = y_n \bigg|_{x = a}
\]

\[
z_{n+1} \bigg|_{x = a} = y_{n+1} \bigg|_{x = a}, \quad z_{n+1} \bigg|_{x = a} = y_{n+1} \bigg|_{x = a} \tag{4}
\]

From these, the following are gained.

\[
C_n = \frac{\sin ak_T}{\sin ak_M} A_n, \quad D_n = \frac{\cos ak_T}{\cos ak_M} B_n, \quad C_{n+1} = \frac{\sin ak_T}{\sin ak_M} A_{n+1}, \quad D_{n+1} = \frac{\cos ak_T}{\cos ak_M} B_{n+1}
\]

Moreover, there exist conditions that displacements and inclination are equal at the dropper.

\[
y_n \bigg|_{x = a} = y_{n+1} \bigg|_{x = a}
\]

\[
T_T \frac{\partial y_{n+1}}{\partial x} \bigg|_{x = a} - T_T \frac{\partial y_n}{\partial x} \bigg|_{x = a} + T_M \frac{\partial z_{n+1}}{\partial x} \bigg|_{x = a} - T_M \frac{\partial z_n}{\partial x} \bigg|_{x = a} = 0 \tag{6}
\]

As a result, the following transfer matrix is gained.

\[
\begin{pmatrix}
A_n \\
B_n
\end{pmatrix} = \frac{1}{T_T k_T \sin 2a k_T + T_M k_M \sin 2a k_M} \begin{pmatrix}
T_T k_T \cos 2a k_T \sin 2a k_M + T_M k_M \sin 2a k_T \sin 2a k_M \\
-T_T k_T \sin 2a k_T \sin 2a k_M - 4T_M k_M \sin 2a k_T \cos 2a k_M \\
T_T k_T \sin 2a k_M \sin 2a k_T + T_M k_M \sin 2a k_T \cos 2a k_M
\end{pmatrix} \begin{pmatrix}
A_{n+1} \\
B_{n+1}
\end{pmatrix} \tag{7}
\]
When this transfer matrix is assigned to $T$, the following relation is obtained.

$$
\begin{pmatrix}
A_n \\
B_n
\end{pmatrix} = T
\begin{pmatrix}
A_{n+1} \\
B_{n+1}
\end{pmatrix}
$$

2.2 Transfer matrix at mast

Similarly, the following solutions are assumed to exist at mast.

$$
\begin{align*}
2.2 & \text{Transfer matrix at mast} \\
\begin{align*}
y_{m2} &= (A_{m2} \sin k_T x + B_{m2} \cos k_T x) e^{-j\omega t} \\
z_{m2} &= (C_{m2} \sin k_M x + D_{m2} \cos k_M x) e^{-j\omega t}
\end{align*}
\end{align*}
$$

$$(9)$$

Where $k_T = \frac{c_T}{\omega}$, $k_M = \frac{c_M}{\omega}$, $c_T = \frac{T_M}{\sqrt{\rho_T}}$, $c_M = \frac{T_M}{\sqrt{\rho_M}}$.

The following boundary conditions also exist at mast.

$$
\begin{align*}
z \big|_{x=a} &= y \big|_{x=a}, & z \big|_{x=-a} &= y \big|_{x=-a}, & z_{m2} \big|_{x=a}, & z_{m2} \big|_{x=0} &= 0
\end{align*}
$$

The following relations are obtained.

$$
\begin{align*}
C &= \frac{\sin a k_T}{\sin a k_M} A, & D &= \frac{\cos a k_T}{\cos a k_M} B \\
C_{m2} &= -\frac{\sin a k_T}{\sin a k_M} A + \frac{\cos a k_T}{\sin a k_M}, & D_{m2} &= 0
\end{align*}
$$

$$(11)$$

Figure 3: A coordinate system at mast

Figure 4: Transfer matrixes at mast
From the equilibrium condition of forces and others, the following matrix is obtained.

\[
\begin{pmatrix}
A_{m2} \\
B_{m2}
\end{pmatrix} = \frac{1}{T_k \kappa_\tau} \begin{pmatrix}
\frac{T_m k_m \sin 2\kappa \tau \cos \kappa_m + T_k \kappa_\tau \cos 2\kappa \tau \sin \kappa_m}{\sin \kappa_m} \\
\frac{T_m k_m \cos 2\kappa \tau \cos \kappa_m - T_k \kappa_\tau \sin \kappa_m \sin 2\kappa \tau}{\sin \kappa_m} \\
\frac{-T_m k_m \cos 2\kappa \tau \cos 2\kappa \tau + T_k \kappa_\tau \sin 2\kappa \tau \sin 2\kappa \tau}{\sin 2\kappa_m} \\
\frac{T_m k_m \sin 2\kappa \tau \cos 2\kappa \tau + T_k \kappa_\tau \cos 2\kappa \tau \sin 2\kappa \tau}{\sin 2\kappa_m}
\end{pmatrix}
\begin{pmatrix}
\mathbf{A} \\
\mathbf{B}
\end{pmatrix}
\]

Provided that this transformation is expressed by \( S_1 \), transformation \( S_2 \) is shown as follows.

\[
S_2 (\mathbf{a}) = S_1^{-1} (-\mathbf{a})
\]

Therefore, the total transformation at mast is

\[
S = S_2 ES_1
\]

\[
S = \begin{pmatrix}
1 \\
1 & -a_1 & a_2
\end{pmatrix} \begin{pmatrix}
a_{11} a_{22} + a_{12} a_{21} & 2 a_{22} \ a_{12} \\
2 a_{11} a_{22} + a_{11} a_{21} & a_{11} a_{22} + a_{12} a_{21}
\end{pmatrix}
\]

Where \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \) are the elements of matrix \( S \).

### 2.3 Coefficient relation in dropper spacing under force

In a dropper spacing the below solutions exist under acting force.

\[
y_{o1} = (A_{o1} \sin \kappa_\tau x + B_{o1} \cos \kappa_\tau x) e^{-j\alpha_1 t}
\]

\[
y_{o2} = (A_{o2} \sin \kappa_\tau x + B_{o2} \cos \kappa_\tau x) e^{-j\alpha_1 t}
\]

\[
z_{o1} = (C_{o1} \sin \kappa_m x + D_{o1} \cos \kappa_m x) e^{-j\alpha_1 t}
\]

\[
z_{o2} = (C_{o2} \sin \kappa_m x + D_{o2} \cos \kappa_m x) e^{-j\alpha_1 t}
\]

Figure 5: A coordinate system in dropper spacing at a forcing point.
Then, the following boundary conditions are given.

\[
\begin{align*}
  y_0 |_{x=-a} &= z_{01} |_{x=-a}, \\
  y_0 |_{x=-a} &= z_{02} |_{x=-a}, \\
  z_0 |_{x=a} &= z_0 |_{x=a}, \\
  \frac{\partial z_0}{\partial x} |_{x=a} &= \frac{\partial z_0}{\partial x} |_{x=a} \\
  y_0 |_{x=a} &= y_0 |_{x=a}, \\
  T_T \frac{\partial y_0}{\partial x} |_{x=a} + F e^{-\frac{\partial}{\partial x} |_{x=a}} &= T_T \frac{\partial y_0}{\partial x} |_{x=a} \\
\end{align*}
\]

From these conditions, the following relations are given.

\[
\begin{align*}
  C_{01} &= \frac{(c_2 - c_1) k_m \cos a k_m}{\Delta}, \\
  D_{01} &= \frac{(c_1 + c_2) k_m \sin a k_m}{\Delta}, \\
  C_{02} &= \frac{(c_2 - c_1) k_m \cos a k_m}{\Delta}, \\
  D_{02} &= \frac{(c_1 + c_2) k_m \sin a k_m}{\Delta}
\end{align*}
\]

Where

\[
\Delta = \begin{vmatrix}
-\sin a k_m & \cos a k_m & 0 & 0 \\
0 & 0 & \sin a k_m & \cos a k_m \\
\sin e k_m & \cos e k_m & -\sin e k_m & -\cos e k_m \\
k_m \cos e k_m & -k_m \sin e k_m & -k_m \cos e k_m & k_m \sin e k_m
\end{vmatrix} = k_m \sin 2 a k_m
\]

\[
c_1 = -A_{01} \sin a k_T + B_{01} \cos a k_T \\
c_2 = A_{02} \sin a k_T + B_{02} \cos a k_T
\]

As a result, the following relation of coefficient is gained.

\[
\begin{align*}
  A_{01} \sin e k_T + B_{01} \cos e k_T &= A_{02} \sin e k_T + B_{02} \cos e k_T \\
  A_{01} \cos e k_T - B_{01} \sin e k_T &= A_{02} \cos e k_T - B_{02} \sin e k_T + \frac{F}{T_T k_T}
\end{align*}
\]

2.4 Transfer matrix between corresponding dropper spacings in different spans
It is necessary to transform the dropper spacing in an arbitrary span to the same position of others span. This transformation of matrix is shown as follows.
\[ \mathcal{R} = T^4 S T' = \left( P \lambda_\tau^{-1} P^{-1} \right)^4 \left( Q \lambda_s^{-1} Q^{-1} \right) \left( P \lambda_\tau^{-1} P^{-1} \right)^4 \]

\[ = P \lambda_\tau^{-4} P^{-1} Q \lambda_s^{-1} Q^{-1} P \left( \lambda T \right)^{-4} P^{-1} \]

\[ = \left( P \lambda_\tau^{-4} P^{-1} Q \right) \lambda_s \left( P \lambda_\tau^{-4} P^{-1} Q \right)^{-1} \left( P \lambda_\tau^{-1} P^{-1} \right)^4 \]

(21)

\[ A_1 A \]

\[ B_3 B, E \]

\[ %=T^4 ST' \]

\[ \left( \begin{array}{c}
A_1 \\
B_3
\end{array} \right) = \mathcal{R} \left( \begin{array}{c}
A_{01} \\
B_{01}
\end{array} \right) \quad \left( \begin{array}{c}
A_{-1} \\
B_{-1}
\end{array} \right) = \mathcal{R} \left( \begin{array}{c}
A_{02} \\
B_{02}
\end{array} \right) \]

(22)

From this, the following relations are gained.

\[ \left( \begin{array}{c}
A_{-1} \\
B_{-1}
\end{array} \right) = \mathcal{R} \left( \begin{array}{c}
A_{02} \\
B_{02}
\end{array} \right) \quad \left( \begin{array}{c}
A_{1} \\
B_{1}
\end{array} \right) = \mathcal{R} \left( \begin{array}{c}
A_{01} \\
B_{01}
\end{array} \right) \]

As the transfer matrix is resolved into an eigen value and eigenvector matrix, the following relation is gained.

\[ \left( \begin{array}{c}
A_1 \\
B_{1}
\end{array} \right) = \left( x_1, x_2 \right) \left( \begin{array}{c}
\lambda_1^n \\
0
\end{array} \right) \left( \begin{array}{c}
\lambda_2^n \\
x_1, x_2
\end{array} \right)^{-1} \left( \begin{array}{c}
A_{01} \\
B_{01}
\end{array} \right) \]

(23)

Where \( x_1 \) and \( x_2 \) are eigenvectors correspond to eigen values \( \lambda_1 \) and \( \lambda_2 \) respectively, and the transfer matrix \( \mathcal{R} \) has a property indifferent to transfer directions. So, the determinant of this matrix is equal to \( 1 \). Therefore, \( \lambda_1 \lambda_2 = 1 \) exists. If \( \lambda_2 > 1 \), it is necessary that \( B_{01}' = 0 \).

\[ \left( \begin{array}{c}
A_{01} \\
B_{01}
\end{array} \right) = \left( \begin{array}{c}
x_1, x_2
\end{array} \right)^{-1} \left( \begin{array}{c}
A_{01} \\
B_{01}
\end{array} \right) \]

(24)
Therefore,

\[
\begin{bmatrix}
A'_{01} \\
0
\end{bmatrix} = \begin{bmatrix} x_1, x_2 \end{bmatrix}^{-1} \begin{bmatrix}
A_{01} \\
B_{01}
\end{bmatrix},
\begin{bmatrix} x_1, x_2 \end{bmatrix} \begin{bmatrix} A'_{01} \\
0
\end{bmatrix} = \begin{bmatrix} A_{01} \\
B_{01}
\end{bmatrix},
\]

\[
A'_{01} x_1 = \begin{bmatrix} A_{01} \\
B_{01}
\end{bmatrix}
\] (25)

The following results are gained.

\[
\begin{bmatrix}
A_n \\
B_n
\end{bmatrix} = A'_{01} \lambda_1^n x_1 \quad (n = 1 \sim \infty)
\]

\[
\begin{bmatrix}
A_{-n} \\
B_{-n}
\end{bmatrix} = B'_{02} \lambda_2^{-n} x_2 \quad (n = 1 \sim \infty)
\] (26)

Also

\[
B'_{02} x_2 = \begin{bmatrix} A_{02} \\
B_{02}
\end{bmatrix}, \quad x_1 = \begin{bmatrix} a_1 \\
b_1
\end{bmatrix}, \quad x_2 = \begin{bmatrix} a_2 \\
b_2
\end{bmatrix}, \quad A'_{01} \begin{bmatrix} a_1 \\
b_1
\end{bmatrix} = \begin{bmatrix} A_{01} \\
B_{01}
\end{bmatrix},
\]

\[
B'_{02} \begin{bmatrix} a_2 \\
b_2
\end{bmatrix} = \begin{bmatrix} A_{02} \\
B_{02}
\end{bmatrix}
\] (27)

From the equations (20), (25) and (27),

\[
\begin{bmatrix}
a_1 \sin e k_T + b_1 \cos e k_T & -a_2 \sin e k_T - b_2 \cos e k_T \\
a_1 \cos e k_T - b_1 \sin e k_T & -a_2 \cos e k_T - b_2 \sin e k_T
\end{bmatrix} \begin{bmatrix}
A'_{01} \\
B'_{01}
\end{bmatrix} = \begin{bmatrix} 0 \\
\frac{F}{T_T k_T}
\end{bmatrix}
\] (28)

The following are gained.


\[ A'_{01} = \frac{F}{T_\tau k_\tau} \left( a_1 \sin e^{k_{\tau}} + b_2 \cos e^{k_{\tau}} \right) \]

\[ B'_{01} = \frac{F}{T_\tau k_\tau} \left( a_1 \sin e^{k_{\tau}} + b_2 \cos e^{k_{\tau}} \right) \]

Ultimately, from the equation (29), the coefficient of corresponding dropper spacing in every span is determined. Now that the coefficients of specified dropper spacing are known, these of other dropper spacings can be easily obtained by the transfer matrix \( T \).

### 3 Calculation Example

Figure 8 shows the mobility of simple catenary equipment by the center of span excited at a forcing point. Therefore, Even number modes should be excited. Figure 8 adequately displays this information. In Figure 8, two peaks can be recognized about 1Hz. They show the resonance in the fundamental modes. The simple catenary equipment vibrates with the same phase in every span at lower frequency, while it does with the reverse phase in every neighboring span at higher frequency. The former frequency is 0.980Hz and the latter is 1.078Hz. Table 1 shows the wire constants of this simple catenary equipment.

<table>
<thead>
<tr>
<th>Term</th>
<th>Tension(N)</th>
<th>Linear density(kg/m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Messenger wire</td>
<td>9,800</td>
<td>0.697</td>
</tr>
<tr>
<td>Contact wire</td>
<td>9,800</td>
<td>0.9877</td>
</tr>
</tbody>
</table>

Span length 50m, 10 droppers at equal spacing in each span

In the range of higher frequencies, two pairs of peak can be also seen. They show the resonance in the third mode, fifth mode, and other odd number modes.
Figure 8: Mobility of simple catenary equipment by a forcing point at a center of span.
Figure 9: (a) Time-history of fundamental vibration mode under the same phase (0.98Hz).

Figure 9: (b) Time-history of fundamental vibration mode under the reverse phase (1.078Hz).
Figure 9 shows the time-history of fundamental vibration mode by a time step one-twelfth of a period $T$. These figures only display the movement of contact wire in the range of ten spans. Also, figure 9 (a) shows the same phase mode, and figure 9 (b) does the reverse phase mode.

Figure 10 shows the motive traces of fundamental vibration mode. This figure is gained by digital simulation. In this digital simulation, the wires are assumed the discrete concentrated masses connected tensioned non flexible strings. As the scale of messenger wire displacement is one forth of that of contact wire. Both the magnitude of messenger wire and contact wire does not accord with mutually.

![Graph showing the time-history of fundamental vibration mode](image)

**Figure 10**: Top, motive traces of fundamental vibration mode under the same phase (0.98Hz), bottom, also those under the reverse phase (1.078Hz)
4 Conclusions

As for the vibrations of simple catenary equipment, the following conclusions are gained.

(1) In the vibrations of simple catenary equipment, two pairs of resonance peaks can be recognized. One is the same phase vibration which contact wire moves in phase in every span, and the other is the reverse vibration which contact wire moves in reverse phase in every neighboring span.

(2) Under the standard simple catenary in Japanese railways, the frequency of fundamental mode under the same phase is 0.980Hz and on the other hand that under the reverse phase is 1.078Hz.

References

