Optimal design of symmetrically laminated plates for maximum buckling temperature
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Abstract

The optimal designs of laminated plates subject to non-uniform temperature distributions are given for maximum buckling temperature. The method of solution involves the finite element method based on Mindlin plate theory and numerical optimisation. A computational approach is developed which involves successive stages of solution for temperature distribution, buckling temperature and optimal fibre angle. Three different temperature loadings are considered and various combinations of simply supported and clamped boundary conditions are studied. The effect of plate aspect ratio on the optimal fibre angle and the maximum buckling temperature is investigated.

1 Introduction

In recent years the use of composite materials in high temperature environments has grown markedly, which has resulted in increased research in thermal loading problems of laminated structures. An important subject in this field is the thermal buckling problem. This field has been studied by various researchers, and a survey paper of developments by Tauchert [1] reviews the recent work in this area.

Laminates subjected to thermal loads may buckle under the action of forces generated by thermal expansion. Results given by Tauchert and Huang [2] indicated that buckling temperatures can be maximised by means of layup optimisation. Optimal design of antisymmetric laminates under thermal loads was given by Adali and Duffy [3] for the non-hybrid and hybrid cases.
In the case of hybrid laminates, the optimisation was carried over the ply angles and a hybridisation parameter, and numerical results are given for simply supported laminates with graphite, boron and glass layers under a uniform temperature change. Multiobjective designs of antisymmetric laminates under thermal loads were given by Adali and Duffy [4] taking the buckling temperature and the maximum deflection as the design objectives. Thangaratnam et al [5] studied the thermal buckling of composite laminated plates using the finite element method. Finite element solutions for the buckling behaviour of laminates subjected to a uniform temperature field were given by Chandrashekhara [6] where transverse shear flexibility was accounted for in the analysis using the thermoelastic version of the first order shear deformable theory.

The present study considers the optimal thermal buckling design of laminated plates with non-uniform temperature distributions and subject to a combination of simply supported and clamped boundary conditions. First the uniform temperature distributions are considered. In addition, it is shown that when the temperature varies linearly across the plate, the critical temperature is approximately double that of the uniform temperature case. Results are presented for different temperature loadings, and for various combinations of clamped and simply supported boundary conditions. The effect of aspect ratio on the optimal buckling temperature and fibre angle is investigated.

2 Basic Equations

Consider a symmetrically laminated rectangular plate of length $a$, width $b$ and thickness $H$ lying in the $x, y, z$ plane and constructed of an arbitrary number $K$ of orthotropic layers of equal thickness $h_k$ and fibre orientation $\theta_k$ where $k = 1, 2, ..., K$. The plate is subject to a thermal loading $T(x, y)$ which is constant in the thickness direction but variable in the $x, y$ directions as shown in Figure 1, where the vertical axis indicates the temperature $T$ at a point $(x, y)$.

The force $(N_x, N_y, N_{xy})$ and moment $(M_x, M_y, M_{xy})$ resultants in a symmetric angle-ply laminate are related to the middle surface displacement components $(u, v, w)$, thermal forces $(N^T_x, N^T_y, N^T_{xy})$ and thermal moments $(M^T_x, M^T_y, M^T_{xy})$ through the constitutive equations:

$$
\begin{align*}
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} &=
\begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
u_x \\
v_y \\
u_x + v_y
\end{bmatrix} -
\begin{bmatrix}
N^T_x \\
N^T_y \\
N^T_{xy}
\end{bmatrix}
\end{align*}
$$

(1)
\[
\begin{bmatrix}
M_z \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
-w_{,xx} \\
-w_{,yy} \\
-2w_{,xy}
\end{bmatrix}
-\begin{bmatrix}
M_x^T \\
M_y^T \\
M_{xy}^T
\end{bmatrix}
\tag{2}
\]

in which a comma denotes differentiation with respect to the subscript and where

\[
A_{ij} = \sum_{k=1}^{K} \bar{Q}_{ij}^{(k)} (z_k - z_{k-1})
\tag{3}
\]

and

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{K} \bar{Q}_{ij}^{(k)} (z_k^3 - z_{k-1}^3)
\tag{4}
\]

are the extensional and bending stiffnesses, respectively, and \(\bar{Q}_{ij}^{(k)}\) are components of the transformed reduced stiffness matrix for the \(k\)-th layer.

The thermal forces and moments appearing in equations (1) and (2) are defined by

\[
\begin{bmatrix}
N_x^T \\
N_y^T \\
N_{xy}^T
\end{bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\alpha_x \\
\alpha_y \\
2\alpha_{xy}
\end{bmatrix} (1, z) T dz
\tag{5}
\]

where \(T\) is the temperature distribution and \(\alpha_x, \alpha_y, \) and \(\alpha_{xy}\) are the coefficients of thermal expansion. For a symmetric laminate exposed to a uniform temperature distribution, say \(T = T_1\), the thermal moments \(M_x^T, M_y^T\) and \(M_{xy}^T\) vanish, while the thermal forces \(N_x^T, N_y^T\) and \(N_{xy}^T\) become functions of \(T_1\) alone.

### 3 Finite Element Formulation

We now consider the finite element formulation of the problem. Let the region \(S\) of the vessel be divided into \(n\) sub-regions \(S_r (S_r \in S; r = 1, 2, \ldots, n)\) such that

\[
\Pi(u) = \sum_{r=1}^{n} \Pi^S_r (u)
\tag{6}
\]

where \(\Pi\) and \(\Pi^S_r\) are potential energies of the vessel and the element, respectively, and \(u\) is the displacement vector. Using the same shape functions
associated with node \( i \) (\( i = 1, 2, \ldots, n \)), \( S_i(x, y) \), for interpolating the variables in each element, we can write

\[
u = \sum_{i=1}^{n} S_i(x, y) u_i, \tag{7}\]

where \( u_i \) is the value of the displacement vector corresponding to node \( i \), and is given by

\[
u = \{u_o^{(i)}, u_o^{(i)}, u_o^{(i)}, \psi_x^{(i)}, \psi_y^{(i)}\}^T. \tag{8}\]

The static buckling problem reduces to a generalised eigenvalue problem of the conventional form, viz.

\[
([K] + \lambda[K_G]) \{u\} = 0 \tag{9}\]

where \([K]\) is the stiffness matrix and \([K_G]\) is the initial temperature matrix. The lowest eigenvalue of the homogeneous system (9) yields the buckling temperature.

### 4 Design Problem and Method of Solution

The objective of the design problem is to maximise the buckling temperature \( T_b \) for a given plate thickness \( H \) by optimally determining the fibre orientations given by \( \theta_k = (-1)^{k+1} \theta \) for \( k \leq K/2 \) and \( \theta_k = (-1)^k \theta \) for \( k \geq K/2 + 1 \). Let the temperature distribution be given by \( T(x, y) = T^t(x, y) \) where \( t(x, y) \) is the temperature distribution corresponding to a unit temperature input. The critical buckling temperature \( T_{cr}(\theta) \) is given by

\[
T_{cr}(\theta) = \min_{m,n}[T_{b,mn}(m, n; \theta)] \tag{10}\]

where \( T_{b,mn} \) is the buckling temperature corresponding to the half-wave numbers \( m \) and \( n \) in the \( x \) and \( y \) directions, respectively. The design objective is to maximise \( T_{cr}(\theta) \) with respect to \( \theta \), viz.

\[
T_{\text{max}} \triangleq \max_{\theta}[T_{cr}(\theta)], 0^\circ \leq \theta \leq 90^\circ \tag{11}\]

where \( T_{cr}(\theta) \) is determined for a given \( \theta \) from the finite element solution of the thermal buckling problem defined by (9).

The present study allows the temperatures to be described along the edges of the plate resulting in a temperature distribution across the plate which is a function of \( x \) and \( y \). Before the buckling problem (9) can be solved, the
temperature distribution has to be determined and this computation is again performed using the heat conduction module of a finite element program. This calculation yields the thermal stress field applicable to that loading condition and this solution is incorporated into the eigenvalue problem (9) in order to compute the corresponding buckling temperature.

The optimisation procedure involves the stages of evaluating the buckling temperature $T_{cr}(\theta)$ for a given $\theta$ and improving the fibre orientation to maximise $T_{cr}$. Thus the computational solution consists of successive stages of analysis and optimisation until a convergence is obtained and the optimal angle $\theta_{opt}$ is determined within a specified accuracy. In the optimisation stage, the Golden Section method is employed.

The overall solution strategy involves three stages of computation and can be summarised as follows:

i) The solution of the temperature distribution problem for given temperatures along the edges by finite elements.

ii) The solution of the thermal buckling problem for a given $\theta_{opt}$ by finite elements.

iii) The solution of the optimisation problem to determine $\theta_{opt}$ corresponding to the maximum buckling temperature by Golden Section method.

This approach allows the solution of the design optimisation problem under a variety of boundary and temperature conditions along the edges.

5 Numerical Results

The structures considered in this study are four-layered symmetrically laminated plates. The material is specified as T300/5208 graphite epoxy for which $E_1 = 181 \text{ GPa}$, $E_2 = 10.34 \text{ GPa}$, $E_{12} = 7.17 \text{ GPa}$ and $\nu_1 = 0.28$. The thermal properties are given as $\alpha_1 = 22.3 \times 10^{-6} \text{ K}^{-1}$, $\alpha_2 = 0.02 \times 10^{-6} \text{ K}^{-1}$, $k_1 = 4.5 \text{ J/m/°K}$, and $k_2 = 0.45 \text{ J/m/°K}$, where $k_1$ and $k_2$ are the coefficients of thermal conductivity in the longitudinal and transverse material directions, respectively.

Three different boundary conditions are implemented along the four plate edges (numbered 1 to 4 in Figure 1). These are (S,S,S,S), (C,S,C,S) and (C,C,C,C) with S representing simply supported and C clamped boundary, while the order refers to edges 1-4, respectively. Rotations around the $x$ and $y$ axes are denoted by $r_x$ and $r_y$, respectively. These conditions may be
explicitly described as follows:

\[(S,S,S,S): u = v = w = r_x = 0 \text{ at } x = 0, a \text{ and } u = w = r_y = 0 \text{ at } y = 0, b.\]

\[(C,C,C,C): u = v = w = r_x = r_y = 0 \text{ at } x = 0, a \text{ and } u = v = w = r_x = r_y = 0 \text{ at } y = 0, b.\]

\[(C,S,C,S): u = v = w = r_x = r_y = 0 \text{ at } x = 0, a \text{ and } u = w = r_y = 0 \text{ at } y = 0, b.\]

The results are given for three different thermal loadings to investigate the effect of temperature distribution on the optimum design and maximum buckling temperature. These loadings can be described as follows:

i) Uniform temperature distribution across the plate.

ii) Linear temperature distribution across the plate.

iii) Nonlinear temperature distribution across the plate.

In all cases, the temperature remains constant through the thickness of the plate, and the results are non-dimensionalised using the following expression:

\[T = \frac{T_{\text{actual}} H \alpha_0}{b}\]

where \(\alpha_0\) is specified as \(1 \text{ K}^{-1}\), \(b = 1\text{m}\) and \(H = 0.01\text{m}\).

Uniform temperature loading

The first case of thermal loading consists of a uniform temperature over the surface of the plate.

Figure 3 shows the graph of the maximum buckling temperature \(T_{\max}\) versus the plate aspect ratio \(a/b\). The maximum buckling temperatures for the clamped plates are seen to be lower than the \((S,S,S,S)\) and the \((C,S,C,S)\) plates. Under mechanical buckling loads, \((C,C,C,C)\) plates tend to give the highest buckling loads. This contrast in the case of temperature loading can be attributed to the fact that simple support conditions provide more degrees of freedom and allow the plate to buckle at higher temperatures. The corresponding optimal fibre angles are shown in Figure 4. It is observed that the boundary conditions have a distinct influence on the optimum fibre orientation.
Linear temperature distribution

Loading two involves a linear variation of the temperature across the plate with the temperature loading along the first plate edge (edge 1, Figure 1) being $T$ and that along the edge 3, $0^\circ$. The resulting temperature distribution is shown in Figure (2a) schematically where the darker shade indicates the higher temperature.

The curves of $T_{\text{max}}$ versus the aspect ratio are shown in Figure 5. It is observed that (S,S,S,S) gives the lowest $T_{\text{max}}$ for $a/b \geq 1$ while under uniform loading (C,C,C,C) gives the lowest $T_{\text{max}}$ for $a/b \geq 1.3$ as seen from Figure 3. The corresponding fibre angles are shown in Figure 6. The trends for $\theta_{\text{opt}}$ are similar to the uniform temperature case (Figure 4) with small changes in the values of $\theta_{\text{opt}}$.

Nonlinear temperature distribution

The third loading case is obtained by setting the temperature of the edge 1 to $T$ and keeping the edge 2 at $0^\circ$. The resulting temperature distribution is shown in Figure (2b). Curves of $T_{\text{max}}$ versus $a/b$ are shown in Figure 7. The corresponding optimal ply angles are shown in Figure 8. It is observed that the sharp increase in $\theta_{\text{opt}}$ in the (C,C,C,C) case is moderate as compared to the previous cases.

6 Conclusions

The optimal thermal buckling design for symmetrically laminated plates was determined. The solutions were obtained using the finite element method in conjunction with an optimisation routine to solve the analysis and design problems, respectively. Results are presented for various temperature loadings and different combinations of boundary conditions.

The effect of optimisation on the buckling load was investigated by plotting the buckling load against the fibre orientation. The optimal ply angles and the corresponding buckling temperatures were given for the aspect ratios $0.5 \leq a/b \leq 2.0$. It was observed that the boundary conditions have a major effect on the optimal ply angle. However, the temperature distributions do not show the same influence on $\theta_{\text{opt}}$. On the other hand, the temperature distribution affects the maximum buckling load considerably.
7 References


Figure 1. Geometry and temperature distribution of the plate.

Figure 2. Temperature fields for a) loading case 2 and b) loading case 3.
Figure 3. $T_{\text{max}}$ versus the aspect ratio $a/b$ (Uniform temperature distribution).

Figure 4. $\theta_{\text{opt}}$ versus the aspect ratio $a/b$ (Uniform temperature distribution).

Figure 5. $T_{\text{max}}$ versus the aspect ratio $a/b$ (Linear temperature distribution).
Figure 6. \( \theta_{\text{opt}} \) versus the aspect ratio \( a/b \) (Linear temperature distribution).

Figure 7. \( T_{\text{max}} \) versus the aspect ratio \( a/b \) (Nonlinear temperature distribution).

Figure 8. \( \theta_{\text{opt}} \) versus the aspect ratio \( a/b \) (Nonlinear temperature distribution).