Global/local approach using hybrid elements for composites
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Abstract

A global/local approach is proposed using partial hybrid elements for composites. The domain of the structure to be analyzed is divided into three areas: local region, global region, and transition region. The 3-D sub-laminated elements are used to predict detail stress distribution in the local region. The degenerated hybrid plate/shell element is used in the global region, and the transition elements are used to connect a degenerated element with the 3-D sub-laminated elements. Because all of the elements in the global/local model are derived from the combined energy variational principle, the global/local finite element model satisfies the continuity of interlaminar stresses at the interlayer surface and traction condition on the top and bottom surface of laminated composites. Two examples are presented to show the efficiency and accuracy of the global/local approach for stress analysis of laminated composites.

1. Introduction

Global/local approach[1] includes such methods as substructuring, zooming[2], domain decomposition[3], s-version[4], variable kinematic element[5]. In numerical analysis, these methods usually consist of conventional displacement elements. For the analysis of composite structures, the main requirement in developing finite element is to satisfy all of the continuity conditions on displacements and transverse stresses at interlaminar surfaces and traction-free condition on the upper and lower surfaces. The conventional displacement elements can not satisfy all of these conditions exactly. In these elements, the transverse stresses are obtained from displacements or in-plane stresses, the accuracy will deteriorate because of the numerical differentiation involved. The conventional hybrid stress elements [6,7] have the ability to satisfy these conditions exactly. In these elements, the stresses are assumed directly from the
beginning and therefore no differentiation on the approximated values has to be carried out. However, the conventional hybrid elements contain six stress components, and this will require much more computer CPU time due to the presence of many stress parameters in assumed stress fields.

For analysis of composite structures, it is not necessary to introduce all components of stresses into an assumed stress field. Within each layer of a laminated composite, all components of displacement, strain and stress are continuous. Furthermore, the in-plane derivatives $\varepsilon_x$, $\varepsilon_y$, $\gamma_{xy}$ and transverse stresses $\sigma_z$, $\tau_{xz}$, $\tau_{yz}$ are also continuous at the layer interface with perfect bonding. This means the in-plane strains and transverse stresses are globally continuous in a laminate, and the other components of strain and stress are at least locally continuous within each layer. In order to enforce the transverse stress continuity, it is only necessary to introduce three transverse stresses into the assumed stress field. Recently, some partial hybrid elements are proposed for analysis of composite structures. They only include three transverse stresses in their assumed stress field and satisfy the continuity condition of the interlaminar stresses. The elements are computationally more efficient than the conventional hybrid element. Huang presented a combined energy variational principle [8]. Using the variational principle, Hoa and co-workers [9,10] developed 3-D partial hybrid finite elements and a multilayer element [11]. Furthermore, they [12-14] developed three other different elements based on the combined energy variational principle. Pian and Li [15] developed two partial hybrid elements based on the mixed variational principles. Jing also developed partial hybrid models which were combined with the higher order plate element [16]. Based on Jing’s element, Wang and Ching proposed a modified partial hybrid stress element [17].

In order to investigate the local effects of composite structures having numerous layers, a global/local approach using partial hybrid elements is proposed in this paper. It uses the hierarchy of mathematical models for different parts of the structure with complicated geometry and takes advantage of the properties of both 3-D solid elements and 2-D laminated elements. In this approach, the domain of the structure to be analyzed is classified into three areas: local region, global region and transition region. In the local region, a 3-D sub-laminated element is used to accurately determine the ply stresses in composite structures near discontinuities. In the global region, a degenerated element is used to model composite structure and to keep the computer storage requirement down. Between the global region and local region, a transition element is used to guarantee the continuity and compatibility of displacement fields in different regions.

2. Element Geometric Matrix and Stiffness Matrix

For the laminated composites, partial stress vectors and partial strain vectors can be defined as
and in considering the effects of discontinuities in properties and orientation of composite materials, the relation between global vector and local vector is

\[
\begin{bmatrix}
\sigma_L \\
-\varepsilon_L
\end{bmatrix} =
\begin{bmatrix}
R_1 & R_2 \\
R_2^T & R_3
\end{bmatrix}
\begin{bmatrix}
\varepsilon_g \\
\sigma_g
\end{bmatrix} =
\begin{bmatrix}
S_1^{-1} & -S_1^{-1}S_2 \\
-S_2S_1^{-1} & S_3-S_2S_1^{-1}S_2
\end{bmatrix}
\begin{bmatrix}
\varepsilon_g \\
\sigma_g
\end{bmatrix}
\]

(2)

where \(R_i\) and \(S_i\) are modulus and compliance components respectively. Within an element, the displacement \(u\) and partial stress \(\sigma_g\) are assumed as

\[
u = [N]\delta \quad \text{and} \quad \sigma_g = [P_g]\beta
\]

(3)

The element partial geometric matrix can be obtained as follows

\[
\begin{bmatrix}
\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}
\end{bmatrix} = [B_g](\delta)^e
\]

(4)

\[
\begin{bmatrix}
\frac{\partial w}{\partial z}, \frac{\partial v}{\partial z}, \frac{\partial w}{\partial y} + \frac{\partial u}{\partial z}, \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y}
\end{bmatrix}^T = [B_L](\delta)^e
\]

(5)

and the element stiffness matrix can be derived from the combined energy variational principle[8],

\[
[K]_e = [K_d] + [K_h]
\]

(6)

\[
[K_h] = [G]^T[H]^{-1}[G]
\]

(7)

\[
[H] = \sum_{j=1}^{N} \int_{V_j} [P_g]^T[R_3]_j[P_g]dV
\]

(8)

\[
[G] = \sum_{j=1}^{N} \int_{V_j} [P_g]^T([B_L] - [R_2]^T[B_g])dV
\]

(9)

\[
[K_d] = \sum_{j=1}^{N} \int_{V_j} [B_g]^T[R_1]_j[B_g]dV
\]

(10)

For the partial hybrid element, the element stiffness matrix consists of a displacement-formulated stiffness matrix and a hybrid-formulated stiffness matrix.
3. 2-D, 8-node Degenerated Plate/Shell Element for the Global Region

The degenerated plate/shell element was originally introduced by Ahmad, Irons and Zienkiewicz [18] for the linear analysis of moderately thick and thin shells. The 2-D degenerated elements in the past work suffer from a common deficiency: constitutive equations lead to discontinuous interlamellar stresses. In order to overcome the stress continuity limitations of single-layer models, Feng and Hoa [12] formulated a degenerated plate/shell element using the combined energy variational principle. In the element, the co-ordinates of a point can be written in the form specified by the 'vector' connecting the upper and lower points and the mid-surface co-ordinates as

\[
\begin{bmatrix}
    X_i \\
    Y_i \\
    Z_i
\end{bmatrix} = \sum_{i=1}^{8} N_i(\xi, \eta) \begin{bmatrix}
    \xi_i \\
    \eta_i \\
    \zeta_i
\end{bmatrix} + \sum_{i=1}^{8} N_i(\xi, \eta) h_i \frac{\zeta}{2} V_{3i}
\]

(11)

\[
V_{3i} = \begin{bmatrix}
    l_{3i} \\
    m_{3i} \\
    n_{3i}
\end{bmatrix} = \frac{1}{h_i} \begin{bmatrix}
    \xi_i \\
    \eta_i \\
    \zeta_i
\end{bmatrix} T \begin{bmatrix}
    \xi_i \\
    \eta_i \\
    \zeta_i
\end{bmatrix}_B
\]

(12)

where, \( h_i \) is the thickness of laminate at node \( i \). The displacement field is assumed as continuous through the entire laminate thickness. It is also assumed that a line that is straight and normal to the middle surface before deformation is still straight, but not necessarily 'normal' to the middle surface after deformation. The displacement throughout the element will be uniquely defined by the three Cartesian components of the mid-surface node displacement \( i \), two rotations of the nodal vector \( V_{3i} \) about orthogonal directions normal to it, and one transverse normal deformation in the thickness direction. The displacement field is

\[
\begin{bmatrix}
    \mathbf{u} \\
    \mathbf{v} \\
    \mathbf{w}
\end{bmatrix} = \sum_{i=1}^{8} N_i \begin{bmatrix}
    u_i \\
    v_i \\
    w_i
\end{bmatrix} + \zeta \begin{bmatrix}
    \phi_x \\
    \phi_y \\
    \psi_z
\end{bmatrix}
\]

(13)

where
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\[ [\phi_i] = \frac{h_i}{2} [V_{1i} \quad -V_{2i} \quad V_{3i}] \]  

(14)

in which, \( V_{1i} \) and \( V_{2i} \) are the unit vectors of the local co-ordinate \((x', y', z')\) at node \( i \). By means of iso-function method [10] and classification of stress modes, the partial stress field is assumed independently as continuous functions along the entire thickness,

\[ \{\sigma_g\} = \{\tau_s\} = [P_g]\{\beta\} = [T][P]\{\beta\} \]  

(15)

The matrix \([T]\) is assumed according to the traction conditions on the upper and lower surfaces of the laminates. The matrix \([P]\) consists of a group of stress modes which can be derived from the assumed displacement field using the iso-function method and the classification method of stress modes.

4. 3-D, 20-node Sub-Laminated Element for the Local Region

Based on the 'sublaminate' concept, a sub-laminated hybrid element can be formulated. In the element constructing process, the combined energy variational principle is treated at sublaminate level. The displacement field is assumed as continuous fields through the entire sublaminate thickness. It is

\[ u = \sum_{i=1}^{20} N_i u_i \quad v = \sum_{i=1}^{20} N_i v_i \quad w = \sum_{i=1}^{20} N_i w_i \]  

(16)

where \( N_i \) is the shape function. The partial stress field is assumed independently as continuous functions along the sublaminate thickness.

\[ \{\sigma_g\} = [P_g]\{\beta\} \]  

(17)

The matrix \([P_g]\) can be derived from the assumed displacement field using the iso-function method and the classification method of stress modes.

5. 15-node Transition Element for the Transition Region

A 3-D transition element was presented[13] based on the 'line node' concept[19]. The 3-D transition element has three line nodes on the interface where it meets the 2-D degenerated plate/shell element and twelve point nodes
on the remaining boundaries where it meets the 3-D elements. The line node can fit any function along the thickness from the degenerated plate/shell element, meanwhile the point nodes have the same polynomial shape functions used for the 3-D element. Hence the displacement compatibility between the transition element and the 3-D solid element or the 2-D plate/shell element is preserved. The element can be used to translate several 3-D solid elements into one 2-D plate/shell element in the thickness of composite structure.

The co-ordinates of any point within the element can be expressed as in the form,

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = \sum_{i=1}^{12} N_i \begin{bmatrix} \xi_i \\ \eta_i \\ \zeta_i \end{bmatrix} + \sum_{a,b}^c N_j \begin{bmatrix} \xi_j \\ \eta_j \\ \zeta_j \end{bmatrix} + \frac{\zeta'}{2} h_j V_{3y}$$

(18)

In which, a, b, and c indicate line nodes. $h_j$ is the length of the plate/shell element along the line node $j$. At the nodes $i = 1-12$, the $N_i$ are the shape functions of the 3-D, 20-node element. At the nodes $i = a, b, c$, the shape functions are

$$N_a = (1 + \xi)(1 + \eta)(\xi + \eta - 1)/4$$

$$N_b = (1 + \xi)(1 - \eta^2)/2$$

$$N_c = (1 + \xi)(1 - \eta)(\xi - \eta - 1)/4$$

(19)

It can be seen that $N_a$, $N_b$ and $N_c$ are same as the shape functions used in the degenerated plate/shell element. The displacements field of the transition element are expressed as follows:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{12} N_i \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix} + \sum_{a,b}^c N_j \begin{bmatrix} u_j \\ v_j \\ w_j \end{bmatrix} + \zeta' \begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix}$$

(20)

Along the boundaries separating subregions, the displacement compatibility is maintained between adjacent elements because $N_i$ are compatible functions. The assumed partial stress field is the same as that in the 3-D, 20-node laminated element.

6. Numerical Example 1 -- Free Edge of a Laminated Strip

To demonstrate the efficiency and accuracy of the global/local model, an analysis of the free edge effect in a laminated strip with the [45/-45/-45/45] sequence is performed under uniaxial extension (X-direction). The laminate has
length of 2L (X-direction), width 2W (Y-direction), and thickness 2H (W = 4H). Each layer is idealized as a homogeneous orthotropic material with the following material coefficients in the principal material coordinate system:

\[
\begin{align*}
E_L &= 20 \times 10^6 \text{ psi} \\
E_T &= 2.1 \times 10^6 \text{ psi} \\
G_{LT} &= G_{TT} = 0.85 \times 10^6 \text{ psi} \\
\nu_{LT} &= \nu_{TT} = 0.21
\end{align*}
\]  

(21)

By means of the symmetry of the problem, the finite element mesh used for analysis is shown in figure 1. For a global/local model, eight elements are used along the thickness near the free edge. In the central region, a degenerated element is used along the thickness. At the boundary between two subregion, eight transition elements are used to connect them. All elements have the same length (=2L). The width of the elements decreases as the free edge is approached. The problem is also analyzed by the layerwise model and conventional 3-D model. The mesh on the X-Y plane in the two models is same as that in the global/local model. The result of interlaminar stress \( \sigma_z \) is shown in figure 2. The stress in the figure has been non-dimensionalized by multiplying it by the factor 20\( \sigma_z/(E_L \varepsilon_0) \), where \( \varepsilon_0 \) is the nominal applied axial strain of \( u_y/(2L) \). The global/local model only takes 62.09 seconds CPU time on VAX 6510 Computer to solve the problem. The layerwise model [1] takes 204.40 seconds CPU time and the 3-D model [1] takes 287.06 seconds CPU time on the same computer. For the analysis, the present global/local model uses 1154 active degrees of freedom totally, the layerwise model uses 2441 active degrees of freedom, and 3-D model uses 2849 active degrees of freedom. It shows that the present global/local model takes less time and uses less active degrees of freedom than other models to solve the same problem and to get the same degree of accuracy.

![Finite Element Mesh for Analysis](image-url)
7. Numerical Example 2 -- A Square Laminate with A Hole

The stress analysis of a square laminate \([45/-45/-45/45]\) with a hole is performed under uniaxial loading (Y-direction). The radius of the hole is \(R\). The laminate has length of \(2L (= 8R)\) and thickness \(2H (= R)\). The material properties are same as the example above. By means of the symmetry of the problem, the finite element mesh used for analysis is shown in figure 3. For a global/local model, four 3-D, 20-node elements are used along the thickness near the hole edge. In the global region, a degenerated element is used along the thickness. Between them, four transition elements are used along the thickness to connect them. The problem is also analyzed by the layerwise model and conventional 3-D model. The mesh on the X-Y plane in the two models is same as that in the global/local model. The result of interlaminar stress \(\sigma_z\) is shown in figure 4. The stress in the figure 4 has been non-dimensionalized by multiplying it by the factor \(\sigma_0 / \sigma_0\), where \(\sigma_0\) is the applied axial stress. The global/local model only takes 61.29 seconds CPU time on VAX 6510 Computer to solve the problem. The layerwise model takes 198.31 seconds CPU time and the 3-D model takes 291.17 seconds CPU time on the same computer. For the analysis, the present global/local model uses 1051 active degrees of freedom totally, the layerwise model uses 2298 active degrees of freedom, and 3-D model uses 2948 active degrees of freedom. It shows that the present global/local model takes less time and uses less active degrees of freedom than other models to solve the same problem and to get the same degree of accuracy.
Figure 3  Finite Element Mesh for Analysis

Figure 4  Interlaminar stress $\sigma_z$ along interply face

8. Conclusion

It has been shown that the present global/local model using partial hybrid element is efficient and accurate for stress analysis of laminated composites. It is a self-consistent global/local finite model in which many types of elements are formulated by using same variational principle combined energy variational principle. The model takes advantage of the capacity of both 3-D elements and 2-D elements. It can be used to model the overall response of
composite structures having numerous layers, and predict local stress distribution such as the interlaminar stresses near the discontinuities of composite structures, with low requirement on computer storage size. The global/local model will be further applied to impact analysis and fracture analysis of laminated composites.

Reference

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