Design synthesis of composite structures through the integration of optimisation techniques and finite element analysis

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Abstract

The present study discusses concepts central to the development of a design synthesis capability for composite structures. By integrating reliable optimisation methods with finite element analysis codes a redesign philosophy based on design variable sensitivity analysis may be implemented. The resultant automated design process therefore relies on decisions based on accurate quantitative measures of the relative performance of the design. The essential concept that is considered here is the efficient representation of the design problem in terms of approximated quantities in order to reduce computational effort without compromising accuracy. Approximation techniques that promote reliability and efficiency are described as is the generation of the design sensitivity data. An example is presented that demonstrates the application of the concepts discussed in the study. The results indicate that the large scale optimisation of composite structures is feasible and will benefit the application of composite materials to high performance structures.

1 Introduction

The use of composite materials in the manufacture of primary components in high performance structural systems has demonstrated the enormous potential of this new class of materials. The high strength to weight and stiffness to weight ratios coupled with the design flexibility offered by fibre
reinforced matrices provides the structural engineer with a material that may, in principle, be tailored to deal with a complex operating specification very efficiently. However the flexibility offered by composites is also largely responsible for the difficulty experienced in the design of high performance structures. The orthotropic nature of the individual layers in the laminate and the discontinuous nature of the composite introduces a large number of design parameters that have to be considered in the design process. This situation results in large and complicated design problems that require a considerable amount of effort when compared against the traditional techniques used for the design of structures in isotropic materials.

The following study discusses the foundations for a design synthesis system for composite structures based on the integration of finite element analysis with an automated non-linear mathematical programming optimisation strategy. The technique is based on the use of analytical behavior sensitivity information generated by finite element analysis which is used to define an approximate model of the design problem.

2 General Problem Formulation

Essentially the optimisation process seeks the selection of specified design variables to achieve, within the constraints placed on the structural behavior or the variables themselves, the optimum goal defined by the objective function. The problem is generally described in a standard format as follows:

Minimise:

\[ F(X) \]  
subject to behavior constraints of the form

\[ G_j(X) \leq 0 \quad j = 1,m \]  
\[ H_k(X) = 0 \quad k = 1,l \]  
and side constraints,

\[ X_i^l \leq X_i \leq X_i^u \quad i = 1,n \]  

where \( F(X) \) represents the objective function, \( G_j(X) \) and \( H_k(X) \) are the behavior constraints and the vector \( X \) contains the values of the \( n \) design variables.

The search for the optimum design is therefore restricted to the confines of the design space defined by the various constraints. A direct search
methodology based on the update formula described by eqn (5) has been used here to search for the optimum design.

\[ X^{k+1} = X^k + \alpha S^k \]  

(5)

The vector \( S \) defines the search direction while \( \alpha \) is the step length moved along the search direction. In most practical design problems the behavior constraints are non-linear with respect to the design variables. The optimisation strategy employed in the solution of the problem must therefore be capable of coping with the characteristics of the design problem if an accurate solution is to be achieved.

3 Mathematical Programming - Method of Feasible Directions

The technique used here to solve the optimisation problem is based on the Method of Feasible Directions developed by Vanderplaats [1]. The method is based on the concept of a push-off factor, \( \theta \), which effectively pushes the search direction away from the tangent to the constraint boundary therefore forcing the search direction into the design space. The feasible search direction is defined in such a way that the design search will follow the an active constraint boundary but will also be allowed to leave the boundary if the direction reduces the objective function more rapidly. The search direction finding problem may then be written:

Maximise :

\[ -\nabla F(X) \cdot S \]  

(6)

Subject to:

\[ \nabla G_j(X) \cdot S + \theta \leq 0 \]  

\[ S \cdot S \leq 1 \]  

(7)  

(8)

The search direction is determined by solving the above problem and a set of dependent variables is chosen. The scalar multiplier is found by executing a one dimensional search with respect to the independent variables thus keeping the design closely associated with the active constraint boundary.

4 Optimisation Strategy

The basic structure of the overall solution strategy used here is based on the generation of an explicit approximate problem statement. This approximation is essentially a high quality simplification of the original optimisation
problem and is termed the sub-problem. The optimiser is then employed to seek the optimum within a move limit envelope. The approximate optimum should, depending on the quality of the approximation, lie relatively close to the real optimum. The convergence is checked between iterations by executing a full finite element analysis. If it is found that the convergence criterion has been satisfied then the process is terminated otherwise a new approximate problem is constructed and the process is repeated. The computational effort is therefore reduced by eliminating the need to execute an expensive finite element analysis whenever response data is required.

5 Approximation Techniques

Several techniques are employed in the generation of the approximate problem such as the reduction of the number of design variables by linking, the temporary deletion of non-critical constraints and the construction of high quality approximations for the retained constraints. Design variable linking effectively reduces the number of independent design variables that need to be considered during the design cycle by eliminating those variables that may be expressed in terms of another variable. Constraint deletion reduces the size of the approximate sub-problem further, this is achieved by examining the value of each of the constraints, if the value falls below some threshold criterion then the constraint is temporarily eliminated from consideration during the design cycle. The approximate functions that are to be representative of the original constraints are generated from Taylor series expansions of the constraints about the present design variable vector, ie.

\[ g_j(X) = g_j^0 + \sum_{i=1}^{n} \frac{\partial g_j}{\partial X_i^0}(X_i - X_i^0) \quad j = 1, \ldots, m_R \]  

where \( m_R \) is the set of retained constraints.

6 Design Variable Sensitivity Analysis

From eqn (9) it is evident that the gradients or sensitivities of the structural responses contained in the constraint approximations are required. This data is obtained by design variable sensitivity analysis which is closely associated with the finite element analysis. It is possible to employ finite difference methods to calculate this information however this approach becomes prohibitively expensive when a large number of independent design variables need to be considered. In this study the gradient information is
acquired through the direct differentiation of the finite element system equation, this effectively yields the nodal displacement gradients that may be used to calculate the gradients of other quantities via constitutive relationships. This technique allows the gradient information to be obtained while the finite element analysis is being executed. This does however require modification of the finite element analysis code.

6.1 Displacement Sensitivities

The global stiffness equation is given by:

\[ K \frac{\partial \mathbf{u}}{\partial X_i} = \frac{\partial \mathbf{P}}{\partial X_i} - \frac{\partial K}{\partial X_i} \mathbf{u} \]  

(10)

where \( K \) is the reduced global stiffness matrix, \( \mathbf{P} \) the reduced load vector and \( \mathbf{u} \) the nodal displacement vector. The stiffness matrix, load and displacement vectors are in general expressed in terms of the design variables \( X \). Differentiation of the system equation with respect to the design variables results in the following:

\[
\frac{\partial \mathbf{u}}{\partial X_i} = K^{-1} \left( \frac{\partial \mathbf{P}}{\partial X_i} - \frac{\partial K}{\partial X_i} \mathbf{u} \right)
\]

(11)

from which the displacement sensitivities may be obtained:

\[
\frac{\partial \mathbf{u}}{\partial X_i} = K^{-1} \left( \frac{\partial \mathbf{P}}{\partial X_i} - \frac{\partial K}{\partial X_i} \mathbf{u} \right)
\]

(12)

The load vector is rarely a function of the design variables and generally the equation may be written in the form:

\[
\frac{\partial \mathbf{u}}{\partial X_i} = K^{-1} \frac{\partial K}{\partial X_i} \mathbf{u}
\]

(13)

The decomposed stiffness matrix is available as a by-product of the displacement solution and therefore the displacement sensitivities may be solved for once the derivative of the reduced global stiffness matrix has been assembled.

6.2 Linearised Approximation of the Tsai-Hill Failure Criterion

The optimisation of composite structures is considered here in terms of weight minimisation subject to failure conditions in each ply of the laminate. The failure criterion employed is the well known Tsai-Hill approach which
states that a ply in a composite laminate is considered to have failed if the magnitude of the failure index exceeds one, i.e.

\[ T_i \leq 1.0 \quad i = 1, N \]  

(14)

where \( N \) refers to the number of layers in the laminate. The Tsai-Hill failure criterion is expressed in terms of the layer stresses as follows:

\[ T_i = \left[ \left( \frac{\sigma_1}{F_1} \right)^2 + \left( \frac{\sigma_2}{F_2} \right)^2 - \left( \frac{\sigma_1 \sigma_2}{F_1^2} \right) + \left( \frac{\sigma_{12}}{F_{12}} \right)^2 \right]^{\frac{1}{2}} \]  

(15)

where \( \sigma_1, \sigma_2 \) and \( \sigma_{12} \) are the ply stresses in the principle material directions and \( F_1, F_2 \) and \( F_{12} \) are the strengths of the ply again referred to the material directions. The constraint based on the Tsai-Hill failure criterion is then given as

\[ G_j = 1 - T_i \geq 0 \]  

(16)

The familiar linearised approximation is

\[ G_j = G_j^0 + \sum_{i=1}^{N} \frac{\partial G_j}{\partial X_i} (X_i - X_i^0) \]  

(17)

where

\[ G_j^0 = 1 - T_i^0 \]  

(18)

\[ \frac{\partial G_j}{\partial X_i} = - \frac{\partial T_i}{\partial X_i} \]  

(19)

Substituting the above equations into the constraint expansion

\[ G_j = 1 - T_i^0 + \sum_{i=1}^{N} \frac{\partial T_i}{\partial X_i} X_i^0 - \sum_{i=1}^{N} N \frac{\partial T_i}{\partial X_i} X_i \]  

(20)

where

\[ \frac{\partial T_i}{\partial X_i} = \frac{1}{2T_i} \frac{2\sigma_1 - \sigma_2 \partial \sigma_1}{(F_1)^2} \frac{1}{2T_i} \left[ \frac{2\sigma_2}{(F_2)^2} - \frac{\sigma_1}{(F_1)^2} \frac{\partial \sigma_2}{\partial X_i} \right] T_i \frac{\partial \sigma_{12}}{\partial X_i} \]  

(22)

7 Calculation of the Stress Gradients

From the above formulation for the gradients of the Tsai-Hill failure indices it is evident that the gradients of the stresses in the material directions are also required. The stress-strain relationship for any orthotropic layer in the composite laminate is given as follows

\[ \{\sigma\} = [Q]\{\epsilon\} \]  

(23)
\([Q]\) is the transformed reduced stiffness matrix for the layer and \(\epsilon\) is the total strain for the laminate which is described in terms of the mid-plane strains and curvatures as follows:

\[
\{\epsilon\} = \{\epsilon^0\} + z\{\kappa\}
\]  

(24)

where \(z\) is the distance measured from the mid-plane to the relevant layer. The derivative of stress-strain equation given by equation 23 is then given by:

\[
\{\frac{\partial \sigma}{\partial X_i}\} = \frac{\partial [Q]}{\partial X_i}\{\epsilon\} + [\overline{Q}]\{\frac{\partial \epsilon}{\partial X_i}\}
\]  

(25)

The strain sensitivity is given by the differentiation of equation , thus

\[
\{\frac{\partial \epsilon}{\partial X_i}\} = \{\frac{\partial \epsilon^0}{\partial X_i}\} + \frac{\partial z}{\partial X_i} + z\{\frac{\partial \kappa}{\partial X_i}\}
\]  

(26)

The mid-plane strain sensitivity is given by:

\[
\{\frac{\partial \epsilon^0}{\partial X_i}\} = [D]\{\frac{\partial u}{\partial X_i}\}
\]  

(27)

and the mid-plane curvature sensitivities by:

\[
\{\frac{\partial \kappa}{\partial X_i}\} = [E]\frac{\partial u}{\partial X_i}
\]  

(28)

where \([D]\) and \([E]\) are the constitutive matrices for the strain and curvature -nodal displacement relationships.

8 Application Example

The following example demonstrates the application of theory described above for the optimisation of a composite structure. Figure 1 shows a composite rectangular plate with a central hole. The plate is simply supported at both the shorter sides allowing rotational displacements at these edges. The loading consists of a normal uniform pressure applied to the edges of the hole resulting in the bending of the plate.

![Figure 1 Composite Rectangular Plate with Central Hole](image)
The laminate is composed of a four layers which are symmetric about the mid-plane. The first and fourth layers have an orientation angle of ninety degrees while the second and third layers were orientated at zero degrees. All the layers are Kevlar in an epoxy matrix for which the elastic moduli or the material directions are 76 and 5.5 Gpa, the shear modulus is 2.3 Gpa. The objective of the optimisation study was to reduce the weight of the structure using the thicknesses of the layers as design variables. Due to the symmetry of the layup there were therefore two design variables, \( h_1 \) and \( h_2 \). The constraints placed on the optimisation presses were stress related in the form of the Tsai-Hill Criterion. Move limits of 50 percent were imposed on the design variables and a convergence criteria of 0.01 was specified for the objective function and design variables.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>Weight [kg]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11.68</td>
</tr>
<tr>
<td>2</td>
<td>8.76</td>
</tr>
<tr>
<td>3</td>
<td>7.01</td>
</tr>
<tr>
<td>4</td>
<td>5.84</td>
</tr>
<tr>
<td>5</td>
<td>4.67</td>
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<tr>
<td>6</td>
<td>4.50</td>
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<tr>
<td>7</td>
<td>4.82</td>
</tr>
<tr>
<td>8</td>
<td>4.81</td>
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</tbody>
</table>

Table 1: Weight reduction iteration history

<table>
<thead>
<tr>
<th>Layer</th>
<th>Initial Thickness</th>
<th>Final Thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 &amp; 4</td>
<td>2.000 mm</td>
<td>0.348 mm</td>
</tr>
<tr>
<td>2 &amp; 3</td>
<td>2.000 mm</td>
<td>1.303 mm</td>
</tr>
</tbody>
</table>

Table 2: Initial and final values of layer thicknesses

Figure 2: Objective Function Iteration History
Eight complete design iterations were required to reduce the weight of the composite laminate from 11.68 kg to 4.82 kg. The iteration history of the objective function is shown in Table 1 and Figure 2. The design variable history is shown in Figure 3 and initial and final values of the layer thicknesses are shown in Table 2. The initial reduction in weight was rapid and was limited by the move limits placed on changes of the design variables. The reduction in the thicknesses of the layers was also restricted by the move limits in iterations one to five, this resulted in the layer thicknesses remaining equal. After the fifth iteration the Tsai-Hill constraint on the second layer dominated the optimisation process as this layer was most sensitive to changes in the design variables therefore causing the divergence of the layer thicknesses.

9 Closure

The foundation for the development of a structural design synthesis capability for composite structures has been presented in this work. The central goal of the methodology described here is the efficient representation of the optimisation problem in order that the optimisation process be reliable and efficient. In essence the approach combines the most sophisticated methods that are available for the two phases of the optimisation process, i.e. response prediction and optimisation. The versatility of finite element analysis allows response prediction to be carried out for a wide range of structural systems. The flexibility of the mathematical programming approach on the other hand describes the optimisation problem in a standard format and
is therefore applicable to most design situations. The seamless integration of these powerful tools therefore represents an opportunity for the development of automated optimisation systems that are required for the design of modern high performance structures.

10 References


