Optimization of structures with correlated eigenstrains

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Abstract

The paper deals with behavior of scalar functional serving for optimization of eigenstrains with stochastically perturbed and correlated values in a laminated cylindrical structure from composite materials. The eigenstrains in layers are required for several reasons, as is improvement of bearing capacity of the laminated cylinder, creep, time delay in construction of large structures, etc. For expressing the relation between stresses and eigenstrains in each layer the transformation field analysis, [3], serves. Recently, in [4] and [5], optimum eigenstrains or optimum eigenstresses were solved under assumption that these quantities had stochastically perturbed and non-correlated values. In this paper the optimum eigenstrains are searched, for which the cost functional similar to that in [4], [5], attains its minimum, but the eigenstrains have stochastically perturbed and correlated values.

1 Introduction

The quality of such a system is usually assessed in view of values of either scalar cost function or another type of cost functional. A part of this functional there are input parameters that may describe both shape and physical properties of the structure. In this paper the eigenstrains are optimized, i.e. the cost functional attains its minimum, and the eigenstrains have stochastically perturbed and correlated values. Such a problem need not deliver one only solution. This is a consequence of the fact that the in-
puts are given with some deviations from the ideal values that are required in the design, and the optimum is attained only with certain probability. The most probable state is deviated from this optimum by a certain values which are dependent on the input properties. Consequently, the statistical characteristics of the cost functional have to be studied. The functional is approximated in the stationary point area by a quadratic hypersurface. This fact leads to a considerably non-gaussian nature of the random part of the functional, which comes over to the Pearson's $\chi^2_n$ process. The distance between the true optimum and the most probable state is increasing with increasing number of input eigenstrains burdened by imperfections. A special form of the cost functional is studied. It represents the minimum variance of stresses of a cylindrical layered composite structure. The random variables (prestrains, eigenstrains) are considered to be mutually correlated and the correlation is expressed by a kind of distance function. Numerical approach to the solution of the problem of optimal distribution of eigenstrains in laminated cylindrical composite structure with correlated eigenstrains is described. It deals with a minimization of the hoop stresses thru the structure. A discussion on examples with correlated eigenstrains is presented.

![Figure 1: Geometry and denotation of the problem](image-url)
2 Laminated body

Let us consider a layered structure which in undeformed state occupies a domain \( \Omega \). The layers are represented by \( \Omega_p, p = 1, \ldots, n \) and \( n \) is a number of layers. The geometry and denotation of the problem is depicted in Fig. 1.

For simplicity assume that there is an external axisymmetric loading \( p = (p_a, p_b) \) along the entire boundary of the structure. The stress \( (\sigma)_i \) at an arbitrary point \( x \) of the domain \( \Omega_i \) may be expressed as a superposition of stress \( (\sigma^P)_i \) at \( x \in \Omega_i \) due to the external loading \( p \) and a linear hull of the volume average eigenstrains \( \langle \mu >_j \) applied in the layers \( \Omega_j, j = 1, \ldots, n \). As the hoop direction has a prevailing importance, in what follows we consider hoop stresses \( \sigma_{\theta\theta} \) and hoop eigenstrains \( \mu_{\theta\theta} \) only in the optimization.

This can be recorded as, see [3]:

\[
(\sigma_{\theta\theta})_i(x) = (\sigma^P_{\theta\theta})_i(x) + \sum_{j=1}^{m} F_{ij}(x) \langle \mu_{\theta\theta} >_j i = 1, \ldots, n .
\]  

(1)

where \( F_{ij} \) is the influence function matrix relating the hoop stress in sub-domain \( \Omega_i \) and the volume average hoop eigenstrain in subdomain \( \Omega_j \).

Substituting (1) to the dual variational principle, one gets

\[
\Phi(<\mu_{\theta\theta}>) = \sum_{i=1}^{n} \int_{\Omega_i} \left[ (\sigma^P_{\theta\theta})_i(x) + \sum_{j=1}^{m} F_{ij}(x) \langle \mu_{\theta\theta} >_j \right] \nabla L_i^T \nabla \left[ (\sigma^P_{\theta\theta})_i(x) + \sum_{j=1}^{m} F_{ij}(x) \langle \mu_{\theta\theta} >_j \right] d\Omega(x).
\]  

(2)

where \( L_i \) is the stiffness of the layer \( i \), which in our study is assumed to be constant throughout the entire structure. After carrying out the integration of the last integral, we get (as the minimum does not change when multiplying the cost functional by a constant, we drop out the material stiffnesses \( L_i \))

\[
\Phi(<\mu_{\theta\theta}>) = \sum_{i=1}^{n} \left[ <\sigma^P_{\theta\theta} >_i + \sum_{j=1}^{m} <F >_{ij} <\mu_{\theta\theta} >_j \right]^2 .
\]  

(3)

and \( <.> \) are volume average quantities. Let us consider the deviations \( u_j \) from \( <\mu_{\theta\theta} >_j \). Then, instead of (3), the functional \( \Phi \) may be recast as

\[
\Phi(<\mu_{\theta\theta} > + u) = \sum_{i=1}^{n} \left[ <\sigma^P_{\theta\theta} >_i + \sum_{j=1}^{m} <F >_{ij} (<\mu_{\theta\theta} >_j + u_j) \right]^2 .
\]  

(4)
Even the deterministic problem need not possess unique solution for \( n < m \) - see [3]. But, for \( n > m \), there exist unique solution of the deterministic problem.

### 3 Statistic properties of the cost functional

Introducing an assumption that the deviations of the input parameters (e.g. thicknesses of layers of a layered structure, elasticity constants, eigenstrains, etc.) from their nominal values are not too large, they are gaussian, centered and moreover, the cost functional in its domain is a continuous and sufficiently smooth function of these parameters. As we study the problem in some distance from the stationary point, the linear approximation is not sufficient and we have to study the original quadratic functional (3). Consequently, the perturbation of these parameters with the gaussian distribution of density corresponds to a non-gaussian distribution of density of probability of the values of the cost functional. This property has to be accepted, especially at the extremum point of the functional, where Gaussian curvature appears to be positive.

In the neighborhood of a selected point (the deterministic optimum solution) \( \mu_{\theta \theta}^0 \), the functional (3) at point \( \mu_{\theta \theta} \) may be expressed in Taylor’s series as,

\[
\Phi(\mu_{\theta \theta}) = \Phi(\mu_{\theta \theta}^0) + \sum_{i=1}^{m} r_i u_i + \sum_{i,j=1}^{m} q_{ij} u_i u_j ,
\]

where \( \Phi(\mu_{\theta \theta}^0) \) - the value of the functional in case of nominal values or the mathematical means of the parameters, in our study deterministic optimal solution.

\( r = |r_i|, q = |q_{ij}| \) - vector or matrix of the parameters of the approximate function.

\[
r_i = \frac{\partial \Phi(\mu_{\theta \theta}^0)}{\partial \mu_{\theta \theta}^0} .
\]

\[
q_{ij} = \frac{1}{2} \frac{\partial^2 \Phi(\mu_{\theta \theta}^0)}{\partial \mu_{\theta \theta}^0 \partial \mu_{\theta \theta}^0} , i,j = 1,...,m ,
\]

\( u = \mu_{\theta \theta} - \mu_{\theta \theta}^0 \) - random deviation of the parameters from the nominal values vector.

\( n \) - the number of layers.

\( m \) - the number of statistical variables (parameters), here number of eigenstrains in \( \mu_{\theta \theta} \).

From (3) and the definition, it follows that

\[
r_i = 2 \sum_{\alpha=1}^{n} < F >_{\alpha i} \left[ < \sigma_{\theta \theta}^0 >_{\alpha} + \sum_{\beta=1}^{m} < F >_{\alpha \beta} < \mu_{\theta \theta}^0 >_{\beta} \right] .
\]
First, we compute the mathematical mean of the functional, i.e. in (5) apply the operator of the mathematical mean $\mathbb{E}\{\cdot\}$:

$$E\{\Phi(<\mu_{\theta\theta}>)\} = \Phi(<\mu^0_{\theta\theta}>) + \sum_{i,j=1}^{m} q_{ij} K_{ij}, \quad (8)$$

where $K_{ij}$ - correlation of the parameters with perturbations; square matrix $m \times m$.

In case of functional (3), we have

$$E\{\Phi(<\mu_{\theta\theta}>)\} = \Phi(<\mu^0_{\theta\theta}>) + \sum_{i=1}^{n} \sum_{k,l=1}^{m} <F>_{ikl} K_{kl} <F>_{il}, \quad (9)$$

Note that if the perturbances of parameters are statistically independent, (9) becomes

$$E\{\Phi(<\mu_{\theta\theta}>)\} = \Phi(<\mu^0_{\theta\theta}>) + \sum_{i=1}^{m} q_{ii} D_{ii} \quad (10)$$

where $D_{ii}$ - diagonal matrix $m \times m$ of parameter dispersions.

If the functional attains its minimum at the point $<\mu^0_{\theta\theta}>$ (deterministic optimum point), each $q_{ii}$ is positive. As, in the same time, the dispersions $D_{ii}$ are also positive, it has to hold:

$$E\{\Phi(<\mu_{\theta\theta}>)\} > \Phi(<\mu^0_{\theta\theta}>) \quad (11)$$

From that it follows that any uncertainty in the parameters may the optimal state just worsen, as the mathematical mean point of the process $\Phi(<\mu_{\theta\theta}>)$ is always greater than its value at the optimum point $<\mu^0_{\theta\theta}>$.

The density of probability of the multidimensional process $\Phi(<\mu_{\theta\theta}>)$ in the neighbourhood of the point $<\mu^0_{\theta\theta}>$ is not symmetric, mainly in the neighbourhood of the extremum (optimum). At this point $r_i = 0, \ i = 1, \cdots n$ and the perturbations can affect the value of the functional only in the positive direction, see (11). In case these disturbances are independent, the random part in (3) is the weighted sum of second powers of values of the input processes. Considering them to be gaussian, the density of probability of $\sum q_{ij} u_i u_j$ is of a type $\chi^2_m$, where $m$ is the number of statistical degrees of freedom.

Let us calculate the second order non-centered moment of the functional $\Phi(<\mu_{\theta\theta}>)$. Apply the operator of mathematical mean to the
second power of the expression (5). After some algebra, respecting that the operators \( \mathbb{E} \{ \cdot \} \) and summation may commute, we get:

\[
\mathbb{E} \{ \Phi^2(<\mu_{\theta\theta}>) \} = \Phi^2(<\mu^0_{\theta\theta}>)+ \\
\sum_{i,j=1}^{m} (r_ir_j + 2\Phi(<\mu^0_{\theta\theta}>))q_{ij}\mathbb{E}\{u_iu_j\} + \\
\sum_{i,j,k=1}^{m} r_{ij}q_{jk}\mathbb{E}\{u_iu_ju_k\} + \sum_{i,j,k,l=1}^{m} \mathbb{E}\{u_iu_ju_ku_l\}.
\]  \( (12) \)

If \( u_i \) are the centered normal processes, it holds:

\[
\mathbb{E}\{u_iu_ju_k\} = 0, \\
\mathbb{E}\{u_iu_ju_ku_l\} = K_{ij}K_{kl} + K_{ik}K_{jl} + K_{il}K_{jk},
\]

so that from (12) we have:

\[
\mathbb{E} \{ \Phi^2(<\mu_{\theta\theta}>) \} = \Phi^2(<\mu^0_{\theta\theta}>) + \sum_{i,j=1}^{m} (r_ir_j + 2\Phi(<\mu^0_{\theta\theta}>))q_{ij}K_{ij} + \\
\sum_{i,j,k,l=1}^{m} q_{ij}q_{kl}(K_{ij}K_{kl} + K_{ik}K_{jl} + K_{il}K_{jk}),
\]  \( (13) \)

and for the functional (3), it follows from (13):

\[
\mathbb{E} \{ \Phi^2(<\mu_{\theta\theta}>) \} = \Phi^2(<\mu^0_{\theta\theta}>) + \\
\sum_{i,j=1}^{n} \left[ 4 \sum_{\alpha=1}^{n} <F>_{\alpha i} \left( <\sigma^p_{\theta\theta}>_{\alpha} + \sum_{\beta=1}^{m} <F>_{\alpha\beta} <\mu^0_{\theta\theta}>_{\beta} \right) + \\ 
* \sum_{\gamma=1}^{n} <F>_{\gamma j} \left( <\sigma^p_{\theta\theta}>_{\gamma} + \sum_{\delta=1}^{m} <F>_{\gamma\delta} <\mu^0_{\theta\theta}>_{\delta} \right) + \\
* 2\Phi(<\mu^0_{\theta\theta}>) \sum_{\alpha=1}^{n} <F>_{\alpha i} <F>_{\alpha j} \right] K_{ij} + \\
* \sum_{i,j,k,l=1}^{n} \left( \sum_{\alpha=1}^{n} <F>_{\alpha i} <F>_{\alpha j} \right) \\
* \left( \sum_{\beta=1}^{n} <F>_{\beta k} <F>_{\beta l} \right) (K_{ij}K_{kl} + K_{ik}K_{jl} + K_{il}K_{jk}).
\]  \( (14) \)
In case that the imperfections are mutually not correlated, (13) becomes,

\[ E\{\Phi^2(<\mu_{\theta\theta}>)\} = \Phi^2(<\mu_{\theta\theta}^0>) + \sum_{i=1}^{m}(r_i^2 + 2\Phi(<\mu_{\theta\theta}^0>)q_{ii}D_{ii} + \sum_{i,j=1}^{m}(q_{ii}q_{jj} + 2q_{ij}^2)D_{ii}D_{jj}. \] (15)

After subtracting the 2nd power of (8) from (13), or (14), the dispersion of the functional \(\Phi(<\mu_{\theta\theta}>)\) is obtained as,

\[ D_{\Phi} = E\{\Phi^2(<\mu_{\theta\theta}>)\} - (E\{\Phi(<\mu_{\theta\theta}>)\})^2 = \sum_{i,j=1}^{m}r_ir_jK_{ij} + \sum_{i,j,k,l=1}^{m}q_{ij}q_{kl}(K_{ik}K_{jl} + K_{il}K_{jk}) = \sum_{i,j=1}^{m} \left[ 4\sum_{\alpha=1}^{n} <F>_{\alpha\alpha} <\sigma_{\theta\theta}^p >_{\alpha} + \sum_{\beta=1}^{m} <F>_{\alpha\beta} <\mu_{\theta\theta}^0 >_{\beta} \right] * \sum_{\gamma=1}^{n} <F>_{\gamma\gamma} \left( <\sigma_{\theta\theta}^p >_{\gamma} + \sum_{\delta=1}^{m} <F>_{\gamma\delta} <\mu_{\theta\theta}^0 >_{\delta} \right) K_{ij} + \sum_{i,j,k,l=1}^{m} \left( \sum_{\alpha=1}^{n} <F>_{\alpha\alpha} <F>_{\alpha\gamma} \right) * \left( \sum_{\beta=1}^{n} <F>_{\beta\beta} <F>_{\beta\gamma} \right) (K_{ik}K_{jl} + K_{il}K_{jk}). \] (16)

If the imperfections are mutually non-correlated, it holds in the neighbourhood of the optimum \(<\mu_{\theta\theta}^0>:\)

\[ D_{\Phi} = 2\sum_{i,j=1}^{m}q_{ij}^2D_{ii}D_{jj}. \] (17)

The factor of asymmetry follows from the third central moment, which can be obtained from the following relation:

\[ S_{\Phi} = E\{\Phi^3(<\mu_{\theta\theta}>)\} - 3E\{\Phi(<\mu_{\theta\theta}>)\}E\{\Phi^2(<\mu_{\theta\theta}>)\} + 2(E\{\Phi(<\mu_{\theta\theta}>)\})^3, \] (18)
Gaussian nature of $u_i$ enables one to express $\mathbf{E}\{u_i u_j u_k u_l u_p u_s\}$ by means of products of the correlation functions:

$$\mathbf{E}\{u_i u_j u_k u_l u_p u_s\} = (K_{kl}K_{ps} + K_{lp}K_{ks} + K_{kp}K_{ls})K_{ij} +\right.$$  
\((K_{jl}K_{ps} + K_{jp}K_{ks} + K_{js}K_{lp})K_{ik} + (K_{jk}K_{ps} + K_{jp}K_{ks} + K_{js}K_{lp})K_{il} +\right.$$  
\((K_{jk}K_{ls} + K_{jl}K_{ks} + K_{js}K_{kl})K_{ip} + (K_{jk}K_{lp} + K_{jl}K_{kp} + K_{jp}K_{kl})K_{is}\right). \tag{20}\)

Sum up the results attained so far, we get

$$S_\Phi = -2\Phi(<\mu^0_{\theta \theta}>) + \sum_{i,j,k,l=1}^m q_{ij}q_{kl}K_{ij}K_{kl} + 3 \sum_{i,j,k,l=1}^m r_{i}r_{j}q_{kl}(K_{ik}K_{jl} +\right.$$  
\(+K_{il}K_{jk}) - 2 \sum_{i,j,k,l,p,s=1}^m q_{ij}q_{kl}q_{ps}(K_{jk}K_{ik} + K_{jk}K_{il})K_{ps} +\right.$$  
\(+ \sum_{i,j,k,l,p,s=1}^m q_{ij}q_{kl}q_{ps}[(K_{lp}K_{ks} + K_{kp}K_{ls})K_{ij} + (K_{jp}K_{ls} + K_{js}K_{lp})K_{ik} +\right.$$  
\+(K_{jp}K_{ks} + K_{js}K_{kp})K_{il} + (K_{jl}K_{ks} + K_{js}K_{kl})K_{ip} +\right.$$  
\+(K_{jl}K_{kp} + K_{jp}K_{kl})K_{is} + (K_{ls}K_{ip} + K_{lp}K_{is})K_{jk}]\right). \tag{21}\)

Assuming both $\Phi(<\mu_{\theta \theta}>)$ and $\partial \Phi(<\mu_{\theta \theta}>) / \partial u_i$ sufficiently smooth, it follows from (21) for mutually non-correlated imperfections in the neighbourhood of the extremum:

$$S_\Phi = -2\Phi(\mu_0) \left( \sum_{i=1}^m q_{ii}D_{ii} \right)^2 + 8 \sum_{i,j,k=1}^m q_{ij}q_{jk}q_{ik}D_{ii}D_{jj}D_{kk}. \tag{22}\)

The coefficient of asymmetry, being usually introduced in the form given e.g. in [1], reads:

$$\gamma_\Phi = S_\Phi / D_\Phi^\frac{3}{2} \tag{23}\)
and, in case of increasing $m$, it tends to a constant. At the same time, its relative value tends to zero, as the measure of asymmetry sinks whenever $m$ grows.

In the similar manner, the coefficient of excess as well as the higher-order coefficients may be stated.

4. Example

A cylindrical layered composite structure in the coordinate system $0r\theta z$ is considered. Our aim is to find out the reasonable distribution of deviations $u_i$ from the optimized eigenstrains in the individual layers. Outer compressive load $p = 200$ MPa, thickness of the structure is 50 cm, outer diameter is 1.5 m, and the structure is constructed from 5 equidistant layers. Stiffness coefficients of the AS4/3501-6(012/903g)$_S$ laminate are: $L_{rr} = 14.240$ GPa, $L_{r\theta} = 5.73$ GPa, $L_{rz} = 6.506$ GPa, $L_{\theta\theta} = 112.847$ GPa, $L_{\theta z} = 5.73$ GPa, $L_{zz} = 49.792$ GPa.

As the hoop stresses possess the prevailing meaning in comparison to the other components of the stress tensor, only these quantities are considered. The hoop eigenstrains are the only arguments of the cost functional.

The generalized plain strain, [3], is assumed in the computation. Using the optimization procedure for the minimization of the cost functional (3) with the hoop eigenstrains applied in the layers 2, ⋯ 5, the average compressive hoop stress in the wall attains the value of 600 MPa and $\Phi(\mu_0) = 180 \times 10^4$ while $\Phi(0) = 180.25 \times 10^4$. The initial average hoop stresses in the layers are computed as,

$$< \sigma_{\theta\theta}^p > = (-623.74 \ -582.66 \ -576.34 \ -592.65 \ -624.61)^T$$

and the resulting optimum eigenstrains are

$$< \mu_{\theta\theta}^0 > = (3.625 \ 4.059 \ 2.384 \ -0.694)^T \times 10^{-4}.$$

The influence matrix is

$$< F > = \begin{bmatrix}
2.777 & 2.427 & 2.223 & 2.126 \\
-8.462 & 2.365 & 2.172 & 2.084 \\
2.179 & -8.718 & 2.234 & 2.149 \\
1.852 & 2.071 & -8.760 & 2.287 \\
1.654 & 1.854 & 2.131 & -8.647
\end{bmatrix} \times 10^4$$

while the coefficients of approximation of the cost functional are given in the following table:
Let us assume that the correlation matrix be given by virtue of the distance function: $K_{ij} = |r_i - r_j|$, where $r_i$ is the radius of the central surface of the layer $i$. Then $K_{ij} = 0.1 K_0 |i - j|$, and $K_0$ is some real (in our example $K_0 = 0.00001$). For this correlation matrix we get the following values:

- mathematical mean according to (8): \( \mathbb{E}\{\Phi(<\mu_{\theta\theta}>)}\} = 175.92 \times 10^4 \); second order non-centered moment according to (13): \( \mathbb{E}\{\Phi^2(<\mu_{\theta\theta}>)}\} = = -140.18 \times 10^9 \); dispersion according to (16): \( D_\Phi = 359.49 \times 10^7 \); factor of asymmetry according to (21): \( S_\Phi = -647.60 \times 10^{13} \); coefficient of asymmetry according to (23): \( D_\Phi = 30.045 \).

Let us assume another correlation matrix: $K_{ij} = \exp |r_i - r_j|$. Then $K_{ij} = K_0 \exp(0.1|i - j|)$, and $K_0 = 0.00001$. For this correlation matrix we get the following values:

- mathematical mean according to (8): \( \mathbb{E}\{\Phi(<\mu_{\theta\theta}>)}\} = 186.92 \times 10^4 \); second order non-centered moment according to (13): \( \mathbb{E}\{\Phi^2(<\mu_{\theta\theta}>)}\} = = 294.17 \times 10^9 \); dispersion according to (16): \( D_\Phi = 385.31 \times 10^8 \); factor of asymmetry according to (21): \( S_\Phi = 552.83 \times 10^{12} \); coefficient of asymmetry according to (23): \( D_\Phi = 0.069 \).

References


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