Analytical solution for a new hybrid double-ellipsoidal heat source in semi-infinite body

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Abstract

Weld pool simulation plays a significant role in thermal stress analysis, residual stress calculation as well as microstructure modeling of welded joints and structures. This paper aims to describe the analytical solution for a new hybrid double-ellipsoidal power density moving heat source in semi-infinite body & its application in weld pool simulation. The solution has been obtained by superimposing the solutions for various parts of the proposed hybrid double-ellipsoidal heat source. The weld pool geometry subjected to various heat source parameters has been calculated and compared with results predicted by the double ellipsoidal heat source and available experimental data. Very good agreement between the predicted weld pool geometry and the measured ones has been obtained. The results also show that the hybrid heat source could give an improved prediction for weld pool depth in comparison with the earlier double ellipsoidal heat source.

1. Introduction

Temperature history of the welded components has a significant influence on the residual stresses, distortion and hence the fatigue behavior of the welded structures. Classical solutions for the transient temperature such as Rosenthal’s solutions [1] which deal with the semi-infinite body subjected to instant point, line or surface heat source, can satisfactorily predict the temperature at a distance far enough from the heat source, but fail to do so at its vicinity. Eagar
and Tsai [2] modified Rosenthal’s theory to include a two dimensional (2-D) surface Gaussian distributed heat source with an uniform distribution parameter and found an analytical solution for the temperature of a semi-infinite body subjected to this moving heat source. Their solution was a significant step forward for temperature prediction at near heat source regions. Even though this 2-D solution using the Gaussian heat sources could predict the temperature at regions closer to the heat source, they are still limited by the shortcoming of the 2-D heat source itself with no effect of penetration. This shortcoming can only be overcome if more general heat sources are to be developed and implemented.

Goldak et al. [3] first introduced the three dimensional (3-D) Double Ellipsoidal moving heat source and used FEM to calculate temperature field of a bead-on-plate. They showed that their 3-D heat source could overcome the shortcoming of the previous 2-D Gaussian model to predict the temperature of the welded joints with much deeper penetration. Nguyen et al. [5] has recently developed a closed form analytical solution for this kind of 3-D heat source and showed that this solution can be used for weld pool geometry prediction [5] and for the calculation of residual stresses in weld-bead-on-plate [6].

In this study, a new hybrid heat source has been introduced in order to match analytical solutions with measured weld pool shape more closely. Analytical solution for this new hybrid heat source in semi-infinite body has been obtained by superposition of the already solved double-ellipsoidal heat sources [5]. Subsequently, the hybrid heat source is used to simulate the weld pool geometry and the results are compared with those generated by double-ellipsoidal heat source and with available experimental data.

2 Analytical solution for Goldak’s heat source

Goldak et. al. [3] proposed a semi-ellipsoidal heat source in which the heat flux is distributed in the Gaussian manner throughout the heat source’s volume. Later, the two different semi-ellipsoids were combined together to give a new heat source called a double ellipsoidal heat source. The heat flux within the semi-ellipsoids is described by two different equations. For a point \((x,y,z)\) located within the first semi-ellipsoid, which represents the front part of the welding arc, the heat flux equation is described as

\[
Q(x, y, z) = \frac{6\sqrt{3}}{a_h b_h c_{hf}} \pi \exp \left( -\frac{3x^2}{c_{hf}^2} - \frac{3y^2}{a_h^2} - \frac{3z^2}{b_h^2} \right)
\]

and for a point \((x,y,z)\) located within the second semi-ellipsoid representing the rear section of the arc as

\[
Q(x, y, z) = \frac{6\sqrt{3}}{a_h b_h c_{hb}} \pi \exp \left( -\frac{3x^2}{c_{hb}^2} - \frac{3y^2}{a_h^2} - \frac{3z^2}{b_h^2} \right)
\]

where \(a_h, b_h, c_{hf}\) and \(c_{hb}\) are the ellipsoidal heat source parameters as described in Fig. 1; \(x, y, z\) are the moving coordinates of the heat source; \(Q(x,y,z)\) is the
heat flux \( Q(x,y,z) \) at a point \((x,y,z)\); \( Q \) is arc heat input \((Q = \eta IV)\); \( V \) and \( I \) are the welding voltage and current, respectively; \( \eta \) is arc efficiency; \( r_f \) and \( r_b \) are proportional coefficients representing heat apportionment in front and at back of the heat source, respectively and \( r_f + r_b = 2 \).

It must be noted here that due to the continuity of the volumetric heat source, the values of \( Q(x,y,z) \) given by Eqs. (1) and (2) must be equal at the \( x = 0 \) plane. From that condition, another constraint is obtained for \( r_f \) and \( r_b \) as \( r_f/c_{hf} = r_b/c_{hb} \). Subsequently, the values for these two coefficients are determined as \( r_f = 2c_{hf}/(c_{hf} + c_{hb}) \); \( r_b = 2c_{hb}/(c_{hf} + c_{hb}) \). It is worth noting here that this double ellipsoidal distribution heat source is described by five unknown parameters: the arc efficiency \( \eta \), and the double ellipsoidal axes: \( a_h, b_h, c_{hf} \) and \( c_{hb} \).

![Fig. 1. Double ellipsoidal power density distributed heat source](image)

The solution for the temperature field of a double-ellipsoidal heat source in a semi-infinite body is based on the solution for an instant point source, which satisfies the following differential equation of heat conduction in fixed coordinate [7]

\[
\frac{dT}{dt'} = \frac{\delta Q}{\rho c} \left[ \frac{4\pi a(t - t')} \right]^{3/2} \exp \left( -\frac{(x - x')^2 + (y - y')^2 + (z - z')^2}{4a(t - t')} \right)
\]

(3)

where \( a \) is thermal diffusivity \((a = k/c\rho)\); \( c \) is specific heat; \( k \) is thermal conductivity; \( \rho \) is mass density; \( t, t' \) are time; \( dT/t' \) is transient temperature due to the point heat source \( \delta Q \) and \((x',y',z')\) is location of the instant point heat source \( \delta Q \) at time \( t' \).

Let us consider the solution of a double-ellipsoidal heat source as a result of superimposing a series of instant point heat sources over the volume of the distributed Gaussian heat source. Substituting Eqs. (1) & (2) for the heat flux at
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a point source into Eq. (3) and integrating over the volume of the heat source, the analytical solution for double ellipsoidal moving heat source with a constant speed $v$ from time $t' = 0$ to time $t' = t$ becomes

$$T - T_0 = \frac{3\sqrt{3Q}}{2\pi^2 \sqrt{\pi}} \int_0^t dt' \frac{A'}{\sqrt{12a(t-t')+a_h^2}} + \frac{B'}{\sqrt{12a(t-t')+b_h^2}}$$

where

$$A' = r_f \exp \left( - \frac{3(x-vt')^2}{12a(t-t')+c_{hf}^2} - \frac{3y^2}{12a(t-t')+a_h^2} - \frac{3z^2}{12a(t-t')+b_h^2} \right)$$

$$B' = r_b \exp \left( - \frac{3(x-vt')^2}{12a(t-t')+c_{hb}^2} - \frac{3y^2}{12a(t-t')+a_h^2} - \frac{3z^2}{12a(t-t')+b_h^2} \right)$$

$T$ is temperature at time $t$ and $T_0$ is initial temperature of a point $(x,y,z)$. More details about the derivation of this solution are described elsewhere [5].

3 A new hybrid double-ellipsoidal heat source

It has been reported in earlier work by Nguyen et. al. [5] that a double-ellipsoidal heat source fails to simulate the finger-tip shape of the weld pool cross-section and underestimates the weld pool depth. In this study, an improved heat source is introduced by superimposing several semi-ellipsoids in such a way that they could formulate a shape similar to the crater of the weld pool as shown in Fig. 2.

Fig. 2. New hybrid double-ellipsoidal heat source
Description of the hybrid heat source

The new hybrid heat source consists of two volumes of the upper & lower semi-double-ellipsoids excluding the overlapping volume. The solution for this new hybrid heat source can be obtained by superimposing the solutions that were already obtained for the individual ellipsoids and will be described below. Let us consider the parameters of the upper semi-double-ellipsoid (USE) are $a_{h1}$, $b_{h1}$, $c_{hf1}$ and $c_{hb1}$, the lower semi-double-ellipsoid (LSE) are $a_{h2}$, $b_{h2}$, $c_{hf2}$ and $c_{hb2}$. Parameters of the third middle semi-ellipsoid (MSE) created by the overlapping zone of the first two semi-double-ellipsoids are $a_{h3}$, $b_{h3}$, $c_{hf3}$ and $c_{hb3}$ as shown in Fig. 2.

Subsequently, the heat flux equation for an internal point $(x,y,z)$ of a front semi-ellipsoid is described as

$$Q_{fi}(x,y,z) = \frac{6\sqrt{3}r_f Q}{\pi\sqrt{\pi} a_{hi} b_{hi} c_{hf}} \exp\left(-\frac{3x^2}{c_{hf}^2} - \frac{3y^2}{a_{hi}^2} - \frac{3z^2}{b_{hi}^2}\right)$$

and of a rear semi-ellipsoid as

$$Q_{bi}(x,y,z) = \frac{6\sqrt{3}r_b Q}{\pi\sqrt{\pi} a_{hi} b_{hi} c_{hb}} \exp\left(-\frac{3x^2}{c_{hb}^2} - \frac{3y^2}{a_{hi}^2} - \frac{3z^2}{b_{hi}^2}\right)$$

where $i = 1$ to 3 for each semi-ellipsoid shown in Fig. 2. As the origin of the second and third semi-ellipsoid is determined by $b_{hx}$ then $b_{hx3} = b_{hx1} - b_{hx}$ as shown in Fig. 2. Furthermore, since the third semi-ellipsoid is created by the overlap of the first two semi-ellipsoids then

$$a_{h3} = a_{h2} = a_{h1} \sqrt{1 - \left(\frac{b_{hx}}{b_{h1}}\right)^2}$$

$$c_{hf3} = c_{hf2} = c_{hf1} \sqrt{1 - \left(\frac{b_{hx}}{b_{h1}}\right)^2}$$

$$c_{hb3} = c_{hb2} = c_{hb1} \sqrt{1 - \left(\frac{b_{hx}}{b_{h1}}\right)^2}$$

As $r_f$ and $r_b$ are proportional coefficients representing heat apportionment in front and back of each semi-ellipsoid, then, using the similar assumption which has been used for the double-ellipsoidal heat source gives

$$r_{f1} + r_{b1} + r_{f2} + r_{b2} - r_{f3} - r_{b3} = 2$$

Again, the condition of continuity of the volumetric heat source must be satisfied for the new hybrid heat source and this results in several additional boundary conditions described below:

1. The heat flux portion given by the equations corresponding to the front parts of the hybrid heat source must be the same as that in the rear parts at the plane $x = 0$. This gives
Equation (11) is valid for any point in plane $x = 0$, hence, applying this for point $(0, 0, b_{he})$ gives

$$\frac{1}{a_{h1}b_{h1}} \exp \left( -\frac{3b_{hx}^2}{b_{h1}^2} \right) \left( c_{hf1} - c_{hbf1} \right) + \frac{1}{a_{h2}b_{h2}} \exp \left( -\frac{3b_{hx}^2}{b_{h2}^2} \right) \left( c_{hf2} - c_{hbf2} \right) = 0$$

or

$$\frac{r_{f1}}{a_{h1}b_{h1}c_{hf1}} \exp \left( -\frac{3b_{hx}^2}{b_{h1}^2} \right) = \frac{r_{f2}}{a_{h2}b_{h2}c_{hf2}} = \frac{r_{f3}}{a_{h3}b_{h3}c_{hf3}}$$

Equation (12) can be rewritten from Eqs. (15) and (16) as

$$\frac{1}{a_{h1}b_{h1}} \exp \left( -\frac{3b_{hx}^2}{b_{h1}^2} \right) \left( \frac{r_{f1}}{c_{hf1}} - \frac{r_{h1}}{c_{hbf1}} \right) = 0$$

or

$$\frac{r_{f1}}{c_{hf1}} = \frac{r_{h1}}{c_{hbf1}}$$

Furthermore, substituting $a_{h2}$, $c_{hf2}$ and $c_{hbf2}$ from Eqs (7), (8) and (9) into (15) and (16), then simplifying gives

$$\frac{r_{f2}}{b_{h2}} = \frac{r_{f3}}{b_{h3}}$$

or

$$\frac{r_{f1}}{b_{h1}} \exp \left( -\frac{3b_{hx}^2}{b_{h1}^2} \right) = \frac{r_{f2}}{b_{h2}} \left( 1 - \frac{b_{hx}^2}{b_{h2}^2} \right)^{-1}$$

$$\frac{r_{h1}}{b_{h1}} \exp \left( -\frac{3b_{hx}^2}{b_{h1}^2} \right) = \frac{r_{h2}}{b_{h2}} \left( 1 - \frac{b_{hx}^2}{b_{h2}^2} \right)^{-1}$$
Taking \( r_{f3} \) and \( r_{b3} \) from Eqs. (18) and (19); \( r_{f2} \) and \( r_{b2} \) from Eqs. (20) and (21) then substituting to Eq. (10) gives

\[
 r_{f1} + r_{b1} = 2 \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{22}
\]

where

\[
 A = \frac{b_{h2}}{b_{h1}} \left( 1 - \frac{b_{h3}^2}{b_{h1}^2} \right) \exp \left( -\frac{3 \cdot b_{hx}^2}{b_{h1}^2} \right)
\]

Subsequently, six unknown variables \((r_{f,i} \text{ and } r_{b,i}) \text{ for } i = 1 \text{ to } 3\) which represent the heat portions in six quarter-ellipsoids composing the new hybrid heat source can be found from 6 boundary condition equations namely Eqs. (17b) and (18) to (22) as

\[
r_{f1} = 2c_{hf1}(c_{hf1} + c_{hb1})^{-1} \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{23}
\]

\[
r_{f2} = 2Ac_{hf1}(c_{hf1} + c_{hb1})^{-1} \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{24}
\]

\[
r_{f3} = 2Ac_{hf1} \left( \frac{b_{h3}}{b_{h2}} \right)(c_{hf1} + c_{hb1})^{-1} \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{25}
\]

\[
r_{b1} = 2c_{hb1}(c_{hf1} + c_{hb1})^{-1} \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{26}
\]

\[
r_{b2} = 2Ac_{hb1}(c_{hf1} + c_{hb1})^{-1} \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{27}
\]

\[
r_{b3} = 2Ac_{hb1} \left( \frac{b_{h3}}{b_{h2}} \right)(c_{hf1} + c_{hb1})^{-1} \left[ 1 + A \left( 1 - \frac{b_{h3}}{b_{h2}} \right) \right]^{-1} \tag{28}
\]

As \( b_{h3} = b_{h1} - b_{h3} \) the new hybrid double-ellipsoidal heat source is entirely described by six geometrical parameters namely \( a_{hi}, b_{hi}, c_{hf1}, c_{hb1}, b_{h2} \) and \( b_{hx} \).

**Solution for the new hybrid heat source**

Using the superposition principle, the solution for the new hybrid heat source is obtained as the sum of heat contributed by the first two ellipsoids less the influence of the third one. Let \( T_1, T_2, T_3 \) be parts of the temperature rise of a point \((x,y,z)\) subjected to upper, lower and middle semi-double-ellipsoid part of the hybrid heat source respectively. Then

\[
 T_i - T_o = \frac{3\sqrt{3Q}}{2\pi \rho \pi \sqrt{\pi}} \int_0^t dt' \frac{\sqrt{12a(t-t') + a_{hi}^2}(12a(t-t') + b_{hi}^2)}{\sqrt{12a(t-t') + c_{hi}^2}} \left( A_i + B_i \right) \tag{29}
\]
where

\[ A_i = r_{hi} \cdot \exp \left( -\frac{3(x - w')^2}{12a(t - t') + c_{hi}^2} - \frac{3y^2}{12a(t - t') + a_{hi}^2} - \frac{3z^2}{12a(t - t') + b_{hi}^2} \right) \]

\[ B_i = r_{hi} \cdot \exp \left( -\frac{3(x - w')^2}{12a(t - t') + c_{hi}^2} - \frac{3y^2}{12a(t - t') + a_{hi}^2} - \frac{3z^2}{12a(t - t') + b_{hi}^2} \right) \]

\( T_i \) is temperature at time \( t \) and \( T_o \) is initial temperature of a point \((x,y,z)\). It is worth noting here that variable \( z \) in Eq. (29) should be replaced by \((z - b_{hi})\) for \( i = 2 \) and \( 3 \). This is due to the fact that the origin of coordinate system of both the second and third semi-double-ellipsoid was moved down by a distance \( b_{hi} \) compared with the first one as shown in Fig. 2. Hence the solution for the new hybrid heat source is obtained as

\[ T - T_o = T_1 + T_2 - T_3 \]  

(30)

where \( T_i \) are determined by Eq. (29) for each part of the semi-double-ellipsoid which belongs to the new hybrid heat source as described above.

### 4 Results and discussion

**Numerical procedure**

In this study, a numerical procedure is applied to calculate the solution for the transient temperature field as described by Eqs. (29) and (30) for the hybrid double semi-ellipsoidal distributed heat source. A Fortran77 computer program is written to facilitate the integral calculation in Eq. (29) and to allow for rapid calculation of geometry of the weld pool based on the assumed melting temperature of 1520°C for steel. Since the solution was obtained for a semi-infinite body, the mirror method which combines the temperature distribution in a plate of infinite thickness and its reflected images was adopted [1]. Using this program, the effects of various heat source parameters \((a_{hi}, b_{hi}, c_{hi}, c_{hl}, b_{hl} \) and \( b_{hx} \)) on the predicted weld pool geometry were investigated. The following material properties for high strength steel were used for the calculation: heat capacity, \( c = 600 \text{ J/kg/°C} \); thermal conductivity, \( k = 29 \text{ J/m/s/°C} \); density \( \rho = 7820 \text{ kg/m}^3 \). The welding parameters used for calculation are voltage, \( U = 26 \text{ V} \); current, \( I = 230 \text{ A} \); welding speed, \( v = 30 \text{ cm/min} \) and arc efficiency, \( \eta = 0.85 \). Due to the length limitation of the CADCOMP'2000 paper, fully obtained numerical results are not reported here but will be presented at the conference and fully described in the upcoming Australasian Welding Journal.
Experimental work
Bead-on-plate specimen was fabricated by using a welding robot and Gas Metal Arc Welding with the following welding parameters: voltage $U = 26$ V, current $I = 230$ A, welding speed $v = 30$ cm/min. Shielding gas of 80% Ar plus 20% CO$_2$ was supplied at 20 l/min. The base material was Japanese high strength steel HT780 and filler material used was MIX-60B. Their mechanical properties are given in Tables 1. The geometry of the weld pool were obtained from the bead-on-plate specimen by means of the metallographs taken of the weld pool shape at the surface of the welded plate and at its transverse cross-section as shown in Fig. 8. The data for the weld pool profiles were measured directly from the metallographs using Adobe Photoshop.

![Top view](image1.png)

![Transverse cross-section](image2.png)

Fig. 3 Geometry of the weld pool

<table>
<thead>
<tr>
<th>Materials</th>
<th>Yield strength $S_y$ (MPa)</th>
<th>Ultimate tensile strength $S_u$ (MPa)</th>
<th>Elongation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>HT-780</td>
<td>821</td>
<td>859</td>
<td>31</td>
</tr>
<tr>
<td>MIX-60B</td>
<td>601</td>
<td>662</td>
<td>28</td>
</tr>
</tbody>
</table>

Model verification
Figures 4(a) and (b) show comparisons between the measured and predicted data of the top view of the weld pool shape on the welded plate and its transverse cross-section. The predicted data were calculated using the following parameters of the heat source which provide the best fit with the measured data: $a_h = 10$ mm, $b_h = 2$ mm, $c_{hf} = 10$ mm, $\eta = 0.85$ and $c_{hb} = 2c_{hf}$ for the double ellipsoidal heat source; $a_{hl} = 8$ mm, $b_{hl} = 2$ mm, $c_{hfl} = 8$ mm, $b_{hl} = 1.75$ mm, $b_{h2} = 6$ mm and $\eta = 0.85$, $c_{hbl} = 2c_{hf}$ for the hybrid heat source. These parameters were selected based on the information of their effect on the weld pool geometry reported earlier. The heat transfer material properties used for the calculation was selected for HT-780 steel based on its ranges [8] and they were the same as reported earlier.
It can be seen from Fig. 4(a) that both the present new hybrid heat source & double ellipsoidal heat source can give very good agreement with the measured data if suitable parameters of the heat source are carefully selected. This means that the predicted model can be calibrated with the experimental data by selecting its heat source parameters and can be used for various simulation purposes. However, Fig. 4(b) shows that both models still fail to predict the complex crater shape of the weld pool in the transverse cross-section. The hybrid heat source model can offer slightly better prediction in term of weld pool depth due to its improved geometry. This means that more work needs to be done to improve the predictability of both models to match the complicated finger tip shape of the weld pool transverse cross-section.

Fig. 4 The weld bead geometry simulation: top view of the weld pool (top) and transverse cross-section (bottom).

5 Conclusions

An analytical solution for the transient temperature of a semi-infinite body subjected to a new hybrid double-ellipsoidal moving heat source has been developed and used for the simulation of the weld pool geometry. Both the numerical and experimental results from this study have shown that the new hybrid heat source could offer reasonably good prediction for the weld pool geometry by adjusting the various heat source parameters.
It also shows greater flexibility for the hybrid heat source to calibrate with the various shapes of the weld pool and it could offer better prediction in terms of weld pool depth when compared with previously developed double-ellipsoidal heat source. However, more work needs to be done to improve the predictability of the model to match with the complicated finger tip shape of the weld pool transverse cross-section. The new hybrid heat source developed in this study has a great potential for use in various simulation purposes such as thermal stress analysis, residual stress calculations and microstructure modelling.

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