Thermo-mechanics of no-tension materials

M. Lucchesi(1), C. Padovani(2) & G. Pasquinelli(2)

1 Dipartimento di Costruzioni, Università di Firenze, Piazza Brunelleschi 6, Firenze, Italy
2 Istituto CNUCE-CNR, Via Santa Maria 36, 56126 Pisa, Italy

Abstract

This paper presents a constitutive equation for no-tension materials in the presence of thermal expansion that accounts for the temperature-dependence of their material's constants. Specifically, under the hypothesis of infinitesimal strains, an explicit expression is given for free energy from which the internal energy, entropy and stress are obtained. Then, the basic equations of the thermo-mechanical equilibrium of a no-tension solid are presented, and we observe that if the strain rate is small, thermo-mechanical uncoupling occurs. Finally, the solution to the equilibrium problem of a circular ring made of a no-tension material subjected to a plane stress under the action of both two uniform radial pressures exerted on the inner and outer boundary and a temperature distribution varying linearly with the radius, is calculated.

1 Introduction

With the aim of modelling the behaviour of masonry constructions, the constitutive equation of materials not withstanding tension has been studied by many authors under isothermal conditions [1]. The infinitesimal strain is assumed to be the sum of a positive semi-definite inelastic part and an elastic part on which the stress, negative semi-definite, depends linearly. Moreover, the stress and the inelastic strain, which can be interpreted as fracture strain, are orthogonal. Thus, one obtains a non-linear hyperelastic material, called masonry-like or no-tension material.

However, there are many engineering problems in which the presence of thermal dilatation must be accounted for; consider for example the influence of thermal variations on stress fields in masonry bridges [2], or the thermo-mechanical behaviour of the refractory materials used in the iron and steel industry [3], and finally, geological problems connected with the presence of volcanic calderas such as that of Pozzuoli [4]. In many such cases the thermal variation during the thermo-mechanical process under examination is so high that the dependence of the material constants on temperature cannot be ignored.

In what follows we set forth a constitutive equation for isotropic no-tension materials under non-isothermal conditions. The theory sketched here and presented in more details in [5] has allowed study of numerical techniques for
solution to the equilibrium problem of masonry-like solids in the presence of thermal loads via the finite element method [6].

In Section 2 we assume that the thermal expansion is a spherical tensor of type $\beta(\theta)I$, where $\beta(\theta)$ is a material function of the temperature $\theta$ and $I$ the identity tensor, and then explicitly define the free energy as a function of both $\theta$ and the measure of strain $A = V - I$, with $V$ the left stretch tensor. In view of the target applications, we assume that the displacement gradient $H$ is small. Moreover, although no limitations are placed on the range of temperature variation, we do assume that $\beta(\theta)$, $\beta'(\theta)$ and $\beta''(\theta)$ are also small. These hypotheses allow us to express free energy as function of the infinitesimal strain $E = \frac{1}{2}(H + H^T)$. Once the free energy has been thus approximated, we can then deduce the internal energy, entropy and stress. We thereby obtain a non-linear elastic material that in the absence of thermal variation, conforms to the definition of masonry-like materials presented in [1]. By assuming the classical Fourier hypothesis for heat flux, the material presented in this paper is completely characterized by five functions of the temperature: Young's modulus, Poisson's ratio, thermal expansion $\beta(\theta)$, conductivity and specific heat. In fact, when these material functions are known, the thermodynamic potentials (and consequently the thermo-mechanical behaviour) of the material is determined. At this point, once the energy equation has been obtained, we are in a position to write the basic equations of the thermoelastic theory for no-tension materials. Just as in the linear elastic case, these equations are: the strain-displacement relation, the equation of equilibrium, the constitutive equations for the stress and the heat flux and the equilibrium energy equation. The system we obtain is coupled because the coefficient of the temperature in the energy equation depends on strain and strain rate [5]. In this paper we assume that the strain rate is small, then the thermoelastic equilibrium equations are uncoupled and can be integrated separately.

Treatment of the theory is fully three-dimensional; plane stress and plane strain problems are considered in [5].

By limiting ourselves to the case of thermo-mechanical uncoupling, it is possible to prove that the equilibrium problem of refractory solids under suitable hypotheses of regularity, has a unique solution in terms of stress [5], a well-established result for no-tension materials under isothermal conditions. Finally, in Section 3, still under the hypothesis of thermo-mechanical uncoupling, we consider a circular ring made of a no-tension material subjected to a plane stress consequent to the action of both two uniform radial pressures acting on the inner and outer boundary and a temperature distribution varying linearly with the radius. We assume that thermal expansion $\beta(\theta)$ and Young's modulus are linear functions of the temperature and the Poisson ratio is nil and we determine the stress field and corresponding fractures in the circular ring.
2 No-tension materials

The aim of this section is to formulate a thermoelastic constitutive theory of isotropic no-tension materials. For the concepts of thermodynamics and thermoelasticity necessary for treatment of the theory developed in this section, we refer to [7] and [8]. We assume that the thermal expansion is the spherical tensor \( \beta(\theta)I \), where \( \beta(\theta) \) is a material function of the temperature \( \theta \). Firstly, we set forth the explicit expression of the free energy as a function of symmetric tensor \( \mathbf{A} \) and temperature \( \theta \in [\theta_1, \theta_2] \). Subsequently, under the hypothesis of infinitesimal strain, we deduce approximate expressions for the free energy function and stress as functions of the symmetric part of the displacement gradient \( \mathbf{E} \) and \( \theta \). Let \( E(\theta), v(\theta) \) and \( \beta(\theta) \) be temperature-dependent material functions, such that

\[
E(\theta) > 0, \quad 0 \leq v(\theta) < \frac{1}{2}, \quad \text{for each} \quad \theta \in [\theta_1, \theta_2], \quad \beta(\theta_0) = 0. \tag{1}
\]

with \( \theta_0 \in [\theta_1, \theta_2] \) as the reference temperature, and let us set \( \gamma(\theta) = \frac{2v(\theta)}{1 - 2v(\theta)} \).

For \( \text{Sym} \) the set of symmetric second-order tensors and \( a_1, a_2, a_3 \) with \( a_1 \leq a_2 \leq a_3 \) the eigenvalues of \( \mathbf{A} \), let us consider the following subsets of \( \text{Sym} \times [\theta_1, \theta_2] \)

\[
\mathcal{R}_1 = \{ (\mathbf{A}, \theta) \mid 2(a_3 - \beta(\theta)) + \gamma(\theta)(\text{tr}\mathbf{A} - 3 \beta(\theta)) \leq 0 \}, \tag{2}
\]

\[
\mathcal{R}_2 = \{ (\mathbf{A}, \theta) \mid a_1 - \beta(\theta) \geq 0 \}, \tag{3}
\]

\[
\mathcal{R}_3 = \{ (\mathbf{A}, \theta) \mid a_1 - \beta(\theta) \leq 0, \quad \gamma(\theta)(a_1 - \beta(\theta)) + 2(1 + \gamma(\theta))(a_2 - \beta(\theta)) \geq 0 \}, \tag{4}
\]

\[
\mathcal{R}_4 = \{ (\mathbf{A}, \theta) \mid \gamma(\theta)(a_1 - \beta(\theta)) + 2(1 + \gamma(\theta))(a_2 - \beta(\theta)) \leq 0, \quad 2(a_3 - \beta(\theta)) + \gamma(\theta)(\text{tr}\mathbf{A} - 3 \beta(\theta)) \geq 0 \}. \tag{5}
\]

Now we are in a position to define the free energy function \( \psi(\mathbf{A}, \theta) \) that in the four regions \( \mathcal{R}_4 \) has the following expressions

\[
\psi(\mathbf{A}, \theta) = \xi(\theta) + \frac{E(\theta)}{2(1 + v(\theta)) \rho} \left\{ \frac{v(\theta)}{1 - 2v(\theta)} (\text{tr}\mathbf{A} - 3 \beta(\theta))^2 + (a_1 - \beta(\theta))^2 + (a_2 - \beta(\theta))^2 + (a_3 - \beta(\theta))^2 \right\}, \quad \text{for} \quad (\mathbf{A}, \theta) \in \mathcal{R}_4. \tag{6}
\]
\[ \psi(A, \theta) = \xi(\theta) , \quad \text{for } (A, \theta) \in \mathcal{R}_2. \quad (7) \]

\[ \psi(A, \theta) = \xi(\theta) + \frac{E(\theta)}{2 \rho} (a_1 - \beta(\theta))^2 , \quad \text{for } (A, \theta) \in \mathcal{R}_3. \quad (8) \]

\[ \psi(A, \theta) = \xi(\theta) + \frac{E(\theta)}{2 (1 - \nu^2(\theta)) \rho} \left\{ (a_1 - \beta(\theta))^2 + (a_2 - \beta(\theta))^2 + 2\nu (a_1 - \beta(\theta))(a_2 - \beta(\theta)) \right\} , \quad \text{for } (A, \theta) \in \mathcal{R}_4. \quad (9) \]

where \( \xi(\theta) \) is a material function which will be specified in the following.

Since we are interested in considering infinitesimal strain, we suppose that there exists \( \delta \in [0, 1) \) such that

\[ \|H\| \leq \delta , \ |\beta'(\theta)| \leq \delta , \ |\beta''(\theta)| \leq \delta , \ |\beta'''(\theta)| \leq \delta , \text{ for each } \theta \in [\theta_1, \theta_2] . \quad (10) \]

where \( H \) is the displacement gradient and \( \| \cdot \| \) is the norm induced by the scalar product \( \cdot \) in the space of second-order tensors, \( \|H\| = (H \cdot H)^{1/2} \). By designating \( E = \frac{1}{2}(H + H^T) \) the infinitesimal strain, from the relation \((I + A)^2 = I + H + H^T + HH^T\) and from \((10)_1\), we immediately deduce that \( E = O(\delta)^{(1)} , \ A = O(\delta) , \ A - E = O(\delta^2) \). Thus, for \( e_1, e_2, e_3 \) the eigenvalues of \( E \) with \( e_1 \leq e_2 \leq e_3 \), within an error of order \( o(\delta^2)^{(2)} \), we have

\[ \psi(E, \theta) = \xi(\theta) + \frac{E(\theta)}{2 (1 + \nu(\theta)) \rho} \left\{ \frac{\nu(\theta)}{1 - 2 \nu(\theta)} (\text{tr}E - 3 \beta(\theta))^2 + (e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + (e_3 - \beta(\theta))^2 \right\} , \quad \text{for } (E, \theta) \in \mathcal{R}_1. \quad (11) \]

\[ \psi(E, \theta) = \xi(\theta) , \quad \text{for } (E, \theta) \in \mathcal{R}_2. \quad (12) \]

\[ \psi(E, \theta) = \xi(\theta) + \frac{E(\theta)}{2 \rho} (e_1 - \beta(\theta))^2 , \quad \text{for } (E, \theta) \in \mathcal{R}_3. \quad (13) \]

---

1 Given a mapping \( B \) from a neighborhood of 0 in \( \mathbb{R} \) into a vector space with norm \( \| \cdot \| \), we write \( B(\delta) = O(\delta) \) if there exist \( k > 0 \) and \( k' > 0 \) such that \( \|B(\delta)\| < k |\delta| \) whenever \( |\delta| < k' \).

2 Given a mapping \( B \) from a neighborhood of 0 in \( \mathbb{R} \) into a vector space with norm \( \| \cdot \| \), we write \( B(\delta) = o(\delta) \) if for each \( k > 0 \) there is \( k' > 0 \) such that \( \|B(\delta)\| < k |\delta| \) whenever \( |\delta| < k' \).
\[
\psi(E, \theta) = \xi(\theta) + \frac{E(\theta)}{2(1 - \nu^2(\theta))\rho} \left\{ (e_1 - \beta(\theta))^2 + (e_2 - \beta(\theta))^2 + 
\right.
\]
\[2\nu (e_1 - \beta(\theta))(e_2 - \beta(\theta)) \right\}, \quad \text{for } (E, \theta) \in \mathcal{R}_4, \quad (14)\]

where, within an error of order \(O(\delta^2)\), regions \(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\) can be considered defined in terms of the eigenvalues of \(E\), instead of those of \(A\).

Within an error of order \(o(\delta)\), we can write

\[
T = T(E, \theta) = \rho \partial_E\psi(E, \theta); \quad (15)
\]

then, in view of expressions (11)-(14), by indicating the normalized eigenvectors of \(E\) as \(q_1, q_2, q_3\), accounting for the relations \(D_E e_i = q_i \otimes q_i\), \(i = 1, 2, 3\), and disregarding terms of order \(o(\delta)\), we get (cp [6])

for \((E, \theta) \in \mathcal{R}_1\),

\[
T(E, \theta) = \frac{E(\theta)}{1 + \nu(\theta)} \left\{ E - \beta(\theta) I + \frac{\nu(\theta)}{1 - 2\nu(\theta)} \text{tr}(E - \beta(\theta) I) I \right\}, \quad (16)
\]

for \((E, \theta) \in \mathcal{R}_2\),

\[
T(E, \theta) = 0, \quad (17)
\]

for \((E, \theta) \in \mathcal{R}_3\),

\[
T(E, \theta) = E(\theta) (e_1 - \beta(\theta)) q_1 \otimes q_1, \quad (18)
\]

for \((E, \theta) \in \mathcal{R}_4\),

\[
T(E, \theta) = \frac{E(\theta)}{1 - \nu^2(\theta)} \left\{ [e_1 - \beta(\theta) + \nu(\theta)(e_2 - \beta(\theta))] q_1 \otimes q_1 + [e_2 - \beta(\theta) + \nu(\theta)(e_1 - \beta(\theta))] q_2 \otimes q_2 \right\}, \quad (19)
\]

where \(0\) is the nil tensor.

It immediately follows from relations (16)-(19) that \(T\) and \(E\) are coaxial; moreover, from the definition of regions \(\mathcal{R}_i\) we deduce that \(T\) is negative semi-definite. For \(I\) the fourth-order identity tensor and \(I \otimes I\) is the fourth-order tensor defined by \(I \otimes I [B] = (\text{tr } B) I\), for all tensor \(B\), let \(D(\theta) = \frac{1 + \nu(\theta)}{E(\theta)} I - \frac{\nu(\theta)}{E(\theta)} I \otimes I\)

the elasticity tensor, positive definite in virtue of (1). Let us now put

\[
E^a(E, \theta) = D(\theta) [T(E, \theta)], \quad E^a(E, \theta) = E - \beta(\theta) I - E^a(E, \theta); \quad (20)
\]
it is easy to prove that $E^a(E, \theta)$ is positive semi-definite and orthogonal to stress $T(E, \theta)$. In absence of temperature variations, the materials characterized by the free energy given in (12)-(15) conforms to the isothermal no-tension material studied in [1] and [2].

Let $\lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}$, $\mu = \frac{E}{2(1 + \nu)}$ and $3 \chi = \frac{E}{1 - 2\nu}$, be the Lamé moduli and the coefficient of volumetric expansion of the material, respectively. From (11)-(14), by taking into account of (6)-(9), we can deduce [6] the expression of the entropy $\eta$ and internal energy $\varepsilon$ which here are omitted.

From the expression of the entropy, disregarding terms of order $O(\delta^2)$, we get the specific heat $C_E$ at constant strain [5]

$$C_E(E, \theta) = \theta \partial_\theta \eta(E, \theta) = -\theta \xi''(\theta).$$

Since the thermodynamic potentials are defined within an arbitrary constant, we assume that they vanish for $E = 0$ and $\theta = \theta_0$; in other words, we suppose that the equalities

$$\xi(\theta_0) = \xi'(\theta_0) = 0$$

hold. From (21), in view of (22), we deduce the following relation

$$\xi(\theta) = \int_{\theta_0}^{\theta} C_E(0, \theta') \, d\theta' - \theta \int_{\theta_0}^{\theta} \frac{1}{\theta'} C_E(0, \theta') \, d\theta',$$

which allows to determine the function $\xi(\theta)$, once the specific heat is known.

In order to complete the system of constitutive equations, we assume the usual relation for heat flux

$$q = -\kappa(\theta) \, g,$$

where $\kappa(\theta) \geq 0$ is the conductivity coefficient and $g$ the spatial gradient of temperature. If we assume

$$\dot{E} = O(\delta),$$

disregarding terms of order $O(\delta^2)$, the energy equation is [5]

$$-\text{div} \, q + \rho \, s = -\rho \theta \xi''(\theta) \, \dot{\theta},$$
with $s$ the heat supply per unit mass and $\rho$ the mass density; thus, thermomechanical uncoupling occurs.

The equilibrium problem for no-tension solids has been studied in recent years and the existence of a solution has been proved solely for a rather restricted class of load conditions. However, the uniqueness of the solution is guaranteed only in terms of stress, in the sense that different displacement and strain fields can correspond to the same stress field. Similar considerations can be made for a no-tension material having the constitutive equation proposed in this section [5].

3 An example

Let us consider a circular ring with inner radius $r_1$ and outer radius $r_2 = 2r_1$ made of a no-tension material and subjected to a plane stress under the action of two uniform radial pressures $p_1$ and $p_2$ acting on the internal and external boundary, respectively. Moreover, the ring is subjected to a temperature distribution $\theta$ depending linearly on the radius $r$ and varying from $\theta_1$ for $r = r_1$ to $\theta_2$ for $r = r_2$. We assume that $\beta(\theta) = \alpha (\theta - \theta_0)$, with $\theta_0$ as the reference temperature and $\alpha$ the linear coefficient of thermal expansion, and that the Poisson's ratio $\nu$ is nil. With regard to Young’s modulus $E$, we assume it to be a linear function of temperature varying from $E_2$ for $\theta = \theta_2$ to $E_1 = \frac{1}{2} E_2$ for $\theta = \theta_1$. The choice of quantities $r_1$, $r_2$, $E_1$, $E_2$ such that $\frac{r_1}{r_2} = \frac{E_1}{E_2} = \frac{1}{2}$ is linked to the fact that, if condition $r_2 E_1 = r_1 E_2$ holds true, then the equilibrium equation of the ring is easily integrable; the procedure set forth here for calculating the solution is independent of the value of $\frac{E_1}{E_2}$.

It is possible to prove that for some values of the constants, the stress field corresponding to a linear elastic material is negative semi-definite, and that the elastic solution is then the solution corresponding to a no-tension material. On the other hand, there exist values of the constants such that the radial stress $\sigma_r$ is purely compressive, and circumferential stress $\sigma_\phi$ is negative starting at $r = r_1$, vanishes at a point internal to the circular ring and becomes positive up to $r = r_2$. Thus, if the material does not withstand tension, the stress field for a linear elastic material does not represent a solution to the equilibrium problem. It is possible to calculate a negative semi-definite stress field equilibrated with the loads which is therefore the solution to the equilibrium problem of the circular ring made of a no-tension material. Such a stress field has components [5].
where 

\[ C_1 = \lambda_2 \frac{\lambda_0^2 \left( \frac{\alpha r_0^2}{5} \theta_2 - \theta_1 \right) + 2(r_2 - r_1) \frac{p_1}{E_2}}{2(r_2 - r_1) \left( \frac{\alpha r_0^2}{5} \theta_2 - \theta_1 \right)} - \lambda_1^2 \left( \frac{\alpha r_0^2}{5} \theta_2 - \theta_1 \right) - 2(r_2 - r_1) \frac{p_0}{E_2} \]

\[ C_2 = \lambda_1 \frac{\lambda_0^2 \left( \frac{\alpha r_0^2}{5} \theta_2 - \theta_1 \right) - 2(r_2 - r_1) \frac{p_1}{E_2}}{2(r_2 - r_1) \left( \frac{\alpha r_0^2}{5} \theta_2 - \theta_1 \right) - 2(r_2 - r_1) \frac{p_0}{E_2}} \]

\[ \lambda_1 = -\frac{1 + \sqrt{5}}{2}, \lambda_2 = -\frac{1 - \sqrt{5}}{2}, p_0 = \frac{r_2 p_2}{r_0}, \text{ and } r_0 \text{ is the unique root belonging to the interval } [r_1, r_2] \text{ of the equation } C_1 r_0^{\lambda_1} + C_2 r_0^{\lambda_2} = 0. \]

The circular region \( \Omega_1 \), with inner radius \( r_1 \) and outer radius \( r_0 \), is entirely compressed and does not contain fractures; in the remaining region \( \Omega_2 \) with inner radius \( r_0 \) and outer radius \( r_2 \), the inelastic radial strain is null and the inelastic circumferential strain is

\[ \varepsilon_\Phi^a(r) = -\frac{\alpha}{2(r_2 - r_1)} \theta_2 - \theta_1 - \frac{2(r_2 - r_1) r_2 p_2}{E_2} \left( \frac{1}{r^2} - \frac{1}{r_0^2} \right) + \]

\[ \left( C_1 r_0^{\lambda_1} + C_2 r_0^{\lambda_2} - \frac{\alpha(\theta_2 - \theta_1)r_0^2}{10(r_2 - r_1)} \right) \frac{1}{r}. \]
$= 1.10^{-5} \, (^\circ \text{C})^{-1}$. The dotted line is the solution for a linear elastic material, while the continuous line represents the solution for a no-tension material. The value of the radius separating the compressed region and the cracked region is $r_0 \approx 1.226 \, \text{m}$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Radial stress $\sigma_r$ vs. $r$.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Circumferential stress $\sigma_\theta$ vs. $r$.}
\end{figure}
Figure 3. Circumferential inelastic strain $e_{\phi}^a$ vs. $r$.

Acknowledgements: The financial support of Progetto Finalizzato "Beni Culturali" and Progetto Policentrico "Meccanica Computazionale" is gratefully acknowledged.

References