Contact mechanics of viscoelastic layered surface

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Abstract

A successful viscoelastic layered contact model has been developed. The single layered linear viscoelastic material is assumed to be perfectly bonded to a rigid substrate and contact a rigid indenter without friction under a step load. Two cases are considered; (1) a compressible layered material with a typical Poisson’s ratio of 0.45 and (2) an incompressible layer with a Poisson’s ratio of 0.5.

This paper highlights the methodology and the results obtained under various conditions.

1 Introduction

In many applications involving contact mechanics, the tribological performance of surfaces can be enhanced by covering them with thin layers of soft materials. The dissimilar elastic properties of these layers to those of the substrate are of particular interest owing to their increasing applications in industry. These include overlay of soft lubricants to hard surfaces, sealing rubbers and viscoelastic load bearing members for instance in the motor vehicle industry. Rubber-lined bearings are used in automotive and other applications. Polymeric layers are also increasingly utilised in robotic and prosthetic applications as soft fingertip tactile devices or as friction enhancers in assortment of grippers.

There are many approaches proposed in the literature to analyse the contact interaction between the soft layer and the indenting object surface. It has usually been assumed in these studies that the contacting surfaces can be modelled as semi-infinite elastic solids, based on the Hertzian contact conditions. However, there are two main limitations with this approach; the nature of the layered
structure and its viscoelastic behaviour. The soft layer should be considered as a viscoelastic material since its stress-strain relations are rate-dependent.

The effect of a finite layer thickness has been considered in a number of studies reported by [1,2,3]. The effect of viscoelasticity has been extended to Hertzian contact conditions by various researchers [4,5], in both cases for semi-infinite solids.

The purpose of this study is to analyse contact mechanics of a model of a rigid sphere, indenting a viscoelastic layered surface. The variation of contact area and pressure distribution with time, resulting from an applied step load, is of particular interest.

2 Mathematical Formulation

The viscoelastic material properties are characterised by creep or relaxation functions, thus taking into account the time-dependent behaviour of the material. The approach is applicable to the situations, where the strain remains small. A standard linear solid model is employed to describe the viscoelastic behaviour of the material. Such a model is an arrangement of springs and dashpots in parallel or in series, as shown in Figure 1.

![Figure 1: The standard linear solid (SLS) three element model.](image)

Boltzman's superposition principle is used in order to sum up the stresses and the strains in a historical manner, thus simulating creep, relaxation, recovery, and other dynamic characteristics of the viscoelastic material. Linear viscoelastic behaviour may be specified in general by either a creep or
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a relaxation model according to Leaderman [6]. In the present analysis, a relaxation model is considered, where:

\[ \sigma(t) = \int_{0}^{t} G(t - \tau) \frac{d\varepsilon(\tau)}{d\tau} d\tau \]  

(1)

where \( \sigma(t) \) is stress at time \( t \) and \( G(t) \) is the time dependent relaxation function, having the dimensions of the elastic moduli. This function represents the viscoelastic properties of the material and specifies the stress response to a unit change of strain \( \varepsilon \) at time \( \tau \). Relaxation functions may be either obtained experimentally or deduced from appropriate spring-dashpot models as proposed by Lee et al [7].

In this study a viscoelastic layered model based on a three element standard linear solid is employed. The viscoelastic layer is assumed to be firmly bonded to a rigid substrate and forms a contact with a rigid indenter under a normal step load. Friction between the viscoelastic layer and the indenter is assumed to be negligible and the layer material to be perfectly isotropic and homogeneous. Furthermore, the contact is assumed to be axisymmetric.

The logical approach to the solution of the viscoelastic contact problem is to replace for the elastic constant the relaxation function in equation (1). Such an approach has been highlighted by Radok [8]. Lee et al [4] have shown that this approach can be applied to the contact problems provided that the applied loading induces an increasing contact area.

2.1 Elastic solutions for layered contacts

According to Johnson [9] for a thin bonded elastic layered contact where the contact radius is in general much larger than the layer thickness, the radial and the circumferential strains \((\varepsilon_r, \varepsilon_\theta)\) are negligible. Thus, the stress component in the z direction (see Figure 2) reduces to:

\[ \sigma_z = (\lambda + 2\mu)\varepsilon_z \]  

(2)

where, the simplified strain-displacement relations in the directions of \( r , \theta \) and \( z \) can be obtained from the general expressions given in references [10] and [11], as follows:

\[ \varepsilon_r = \frac{\partial u}{\partial r} \, , \, \varepsilon_\theta = \frac{u}{r} \, , \, \varepsilon_z = \frac{\partial w}{\partial z} \]  

(3)

and the Lame’s constants \((\mu, \lambda)\) are defined as:

\[ \mu = \frac{E}{2(1+v)} \quad \text{and} \quad \lambda = \frac{EV}{(1+v)(1-2v)} \]  

(4)

Furthermore, the compressive strain in the element for the compressible layer is given by the geometry of deformation:

\[ \frac{\partial w}{\partial z} = -\frac{1}{d} \left( \nu - \frac{r^2}{2R} \right) \]  

(5)
where, \( d \) is the layer thickness and \( v_* \) is the maximum penetration depth at \( r = 0 \), which can be determined from the boundary condition where the contact pressure diminishes (i.e. at the end of the contact).

Combining equations (2) to (5) and noting that the contact pressure \( p(r) = -\sigma_z \), the solution can be obtained for a compressible material with a typical value of Poisson’s ratio (\( v \)) less than 0.5 as [3]:

\[
p(r) = \frac{Ea^2(1-v)}{2Rd(1-2v)(1+v)} \left[ 1 - \left( \frac{r}{a} \right)^2 \right] 
\]

For an incompressible layered material with a Poisson’s ratio of 0.5, the pressure distribution has been obtained by reference [3] as:

\[
p(r) = \frac{Ea^4}{32Rd^3} \left[ 1 - \left( \frac{r}{a} \right)^2 \right]^2 .
\]
2.2 Viscoelastic solutions for layered contacts

For the viscoelastic material, the contact radius and the pressure distribution are time dependent. Therefore, it is feasible to follow approach highlighted in reference [8], incorporating time in equations (6) and (7). The contact radius can be obtained by considering the load balance:

\[ W = 2\pi \int_0^{a(t)} rp(r, t) \, dr \]  

(8)

Substituting the contact pressure from equation (6) for the compressible layered contact into the above equation:

\[ k(t) W = \frac{1 - \nu}{(1 + \nu)(1 - 2\nu)} \frac{E\pi}{4Rd} a^4(t) \]

(9)

where the relaxation function \( k(t) \) for an incompressible, three-parameter viscoelastic solid is given as: [5]

\[ k(t) = \frac{1}{4} \left[ \frac{1}{E_*} e^{-\frac{-t}{\tau_2}} + \frac{1}{E_\omega} \left( 1 - e^{-\frac{-t}{\tau_2}} \right) \right] H(t) \]

(10)

It should be noted that the above relaxation function is also used for the compressible layered contact in this study. For the case of an elastic incompressible material, \( k(t) \) becomes independent of time and is given: [10]

\[ k(t) = \frac{1}{4E_*} \] for all values of \( t \).

The relaxation function \( k(t) \) characterises the material properties of the viscoelastic body, \( \tau_2 \) is the relaxation time, \( E_* \) is the instantaneous elastic modulus and \( E_\omega \) is the asymptotic elastic modulus, and \( H(t) \) is the unit step function.

The radius of the contact region is, thus:

\[ a^2(t) = \left[ \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)} \frac{RdW}{E_*\pi} \left( 2 - e^{-\frac{-t}{\tau_2}} \right) \right]^{\frac{1}{2}} H(t) \]

(11)

where, the asymptotic elastic modulus is assumed to equal to half the instantaneous elastic modulus in equation (10) [5]. Equation (11) can also be written as:

\[ a^2(t) = a_*^2 \left( 2 - e^{-\frac{-t}{\tau_2}} \right)^{\frac{1}{2}} H(t) \]

(12)

where, the instantaneous radius of contact is given by:

\[ a_*^2 = \left[ \frac{(1 + \nu)(1 - 2\nu)}{(1 - \nu)} \frac{RdW}{E_*\pi} \right]^{\frac{1}{2}} \]

(13)

The corresponding pressure distribution for the compressible viscoelastic material can be derived as follows:
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\[ k(t) * p(r, t) = \frac{(1 - v)}{(1 + v)(1 - 2v)} \frac{E_o}{2Rd} (a^2 - r^2) H(t) \]  

(14)

where, the symbol ‘*’ denotes the convolution integral. This requires inverting the convolution on the right hand side and taking Laplace Transform. The pressure distribution can be written in the following form:

\[ p(r, t) = \frac{(1 - v)}{(1 + v)(1 - 2v)} \frac{1}{2Rd} \left[ R_g * (a^2 - r^2) H(t) \right] \]

(15)

where

\[ R_g = 2E_o \left( 1 + e^{-\frac{t}{\tau_2}} \right) \]

(16)

and

\[ p(r, t) = \frac{(1 - v)}{(1 + v)(1 - 2v)} \frac{E_o}{4Rd} \left[ 1 + e^{-\frac{t}{\tau_2}} \right] \frac{\partial}{\partial \tau} \left[ (a^2 - r^2) H(\tau) \right] d\tau \]

(17)

Equation (17) can be further simplified as: (using integration by parts)

\[ p(r, t) = \frac{E_o a^2 (1 - v)}{8Rd(1 + v)(1 - 2v)} \left[ e^{-\frac{t}{\tau_2}} \left[ 1 - \left( \frac{r}{a_o} \right)^2 \right] + \left[ 2 - e^{-\frac{t}{\tau_2}} \right] \left( \frac{r}{a_o} \right)^2 \right] \]

\[ \left[ 2 - \left( \frac{r}{a_o} \right)^2 \right] \int_{f(0)}^{f(t)} d\xi \]

(18)

where: \( f(t) = (2 - e^{-\frac{t}{\tau_2}})^{\frac{1}{2}} - \left( \frac{r}{a_o} \right)^2 \). For the case of an incompressible viscoelastic layer with a Poisson’s ratio of 0.5, the pressure distribution and the contact radius can be derived in a similar manner as in the case of the compressible material. The contact radius can be easily derived from equations (7), (8) and (10):

\[ a^2(t) = \left[ \frac{24d^3RW}{\pi E_o} \left( 2 - e^{-\frac{t}{\tau_2}} \right) \right]^{\frac{1}{6}} H(t) \]

(19)

or written as:

\[ a^2(t) = a_o^2 \left[ 2 - e^{-\frac{t}{\tau_2}} \right]^{\frac{1}{6}} H(t) \]

(20)

where, the elastic contact radius is given by:

\[ a_o^2 = \left( \frac{24d^3RW}{\pi E_o} \right)^{\frac{1}{2}} \]

(21)

The corresponding pressure distribution can be derived as follows:

\[ k(t) * p(r, t) = \frac{E}{32d^3R} \left( a^2 - r^2 \right)^2 H(t) \]

(22)
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which can be simplified as:

$$p(r, t) = \frac{1}{32d^3R} \left( R_s (a^2 - r^2)^2 \right) H(t)$$  \hspace{1cm} (23)

Therefore, the contact pressure for the viscoelastic layered material with a Poisson’s ratio of 0.5 is given as:

$$p(r, t) = \frac{E_o a^4}{16 d^3 R} \left\{ e^{-\frac{r}{d}} \left( 1 - \left( \frac{r}{a_s} \right)^2 \right)^2 + \left[ 2 - e^{-\frac{r}{d}} \right] \left[ \left( \frac{r}{a_s} \right)^2 \right] \right\}$$  \hspace{1cm} (24)

where:

$$f(t) = \left[ (2 - e^{-\frac{r}{d}}) \left( \frac{r}{a_s} \right)^2 \right]$$

3. Results and Discussion

The pressure distribution is normalised with respect to its maximum instantaneous initial value at $t = 0$ as shown in Figures 3 and 4 for a compressible material with a typical Poisson’s ratio of 0.45 and for an incompressible material with a Poisson’s ratio of 0.5 respectively. It is interesting to note that the instantaneous pressure distributions conform to their corresponding elastic solutions given by equations (6) and (7). This fact corroborates the numerical scheme adopted in this study. As time progresses, the maximum pressure falls due to viscoelastic relaxation of the layered material. There is a corresponding increase in the contact footprint radius. This effect is more pronounced for the case of compressible bonded layers as expected. For example, in the case of the compressible material, the maximum pressure is decreased by about 30% during a period equivalent to 10 times the duration of the material’s relaxation whilst for an incompressible layer the corresponding decrease is about 23%. It is also interesting to note that the difference in the pressure distribution profiles between the compressible and the incompressible materials. This can be explained from equations (6) and (7), that in the case of the incompressible material, the pressure gradient diminishes towards the contact extremities.

It should be realised that the above results are presented in normalised coordinates in order to provide general solutions. In practice, the maximum pressures are an order of magnitude higher for incompressible layers, whilst the corresponding footprint area is several times smaller. It should also be noted that the present analysis for the incompressible layered model can be applied for most elastomeric materials such as rubber with a Poisson’s ratios close to 0.5, and the
compressible layered model for polymers with a Poisson’s ratio in the range between 0.4 and 0.45.

4. Conclusion

The axisymmetric contact of a perfectly bonded isotropic, homogeneous, viscoelastic layer bonded to a rigid substrate of finite thickness, and contacting a frictionless rigid indenter under a constant step load has been examined in the present study. Non-dimensional general solutions of the contact radius and the contact pressure have been obtained for both the compressible viscoelastic layered contact model and the incompressible model. The main applications of the present results are in elastomeric and polymeric coatings.

REFERENCES

Nomenclature

\( a \) Contact radius for viscoelastic material
\( a_* \) Contact radius for elastic material
\( d \) Layer thickness
\( E, E_* \) Elastic moduli
\( G(t) \) Relaxation function in shear
\( k(t) \) Relaxation function in dilatation
\( p \) Pressure distribution
\( R \) Radius of curvature
\( R_{g}(t) \) Kernel for pressure
\( t \) Time
\( u, w \) Displacements in \( x, \) and \( z \) directions
\( v_* \) Maximum depth of penetration
\( W \) Applied load
\( \varepsilon_z \) Normal strain
\( (r, \theta, z) \) Cylindrical coordinates
\( \sigma_z \) Normal stress
\( \tau_2 \) Relaxation time
\( \xi \) Substituted variable
Figure 3: The transient pressure distribution for a rigid sphere indenting a three parameter compressible layer material with a Poisson's ratio of 0.45 subjected to a step load.

Figure 4: The transient pressure distribution for a rigid sphere indenting a three parameter incompressible layer material with a Poisson's ratio of 0.5 subjected to a step load.