Application of BEM to non-conforming elastoplastic contact problems

D. Martin, M.H. Aliabadi
Wessex Institute of Technology, Ashurst Lodge, Ashurst, Southampton, SO40 7AA, UK

Abstract - In this paper, a boundary element formulation for non-conforming contact problems, with the consideration of elastoplasticity, is presented. In order to solve the contact problem, a direct constraint technique is employed. Friction between the bodies is taken into account, and the materials may have different elastoplastic properties. An initial strain BEM formulation is used to study the elastoplastic problem. The material is assumed to obey the Von Mises yield criterion with its associated flow rule. An incremental loading technique is proposed, which enables to follow accurately the loading history of the problem. An example of a cylinder on a flat foundation is presented.

Introduction

A contact problem is of the non-conforming type, if the surfaces of the bodies coming into contact do not have the same shape in the unloaded state. In the presence of friction, non-conforming contact problems are non linear, even with the assumption of linear elasticity. These problems are also load dependent, due to the irreversibility of frictional phenomena. The Boundary Element Method (BEM) has been applied to a wide range of elastic, frictional contact problems, (see refs. [1] - [4]). The extension to elastoplastic materials has also been reported (refs. [5],[6] and [7]), although only conforming geometries have been dealt with. In this paper, non-conforming, frictional elastoplastic contact problems are, for the first time, solved. The direct constraint technique and the initial strain approach for the elastoplastic BEM formulation [8],[9] are employed. The material is assumed to obey the Von Mises yield criterion with its associated flow rule [10]. The model can handle either perfectly plastic or work hardening materials. The contact areas are modelled using linear boundary elements, quadratic elements are used on all other boundaries.
Incremental boundary integral equation

In order to follow the loading path accurately, an incremental application of the load is needed. In the first place, the governing equations will be written in an incremental way.

As a starting point, it is recalled the displacement integral equation, which takes into account the elastoplasticity (see ref. [9]):

\[ c_{ij} u_j + \int_{\Gamma} p_{ij}^* u_j d\Gamma = \int_{\Gamma} u_{ij}^* p_j d\Gamma + \int_{\Omega} \sigma_{ijk}^* \varepsilon_{jk}^p d\Omega. \]  

Equation (1) describes a body of domain \( \Omega \), and boundary \( \Gamma \), which due to external actions has undergone displacements \( u_j \), tractions \( p_j \) and plastic strains \( \varepsilon_{jk}^p \). These fields are equilibrated, and compatible with the boundary conditions of the problem. If they are supposed known, as a result of a previous analysis, and an increment of the external loads, \( \Delta P \), is considered, all tractions, displacements and plastic strains will be perturbed, such that the following expression will hold:

\[ c_{ij} (u_j + \Delta u_j) + \int_{\Gamma} p_{ij}^* (u_j + \Delta u_j) d\Gamma = \int_{\Gamma} u_{ij}^* (p_j + \Delta p_j) d\Gamma + \]

\[ + \int_{\Omega} \sigma_{ijk}^* (\varepsilon_{jk}^p + \Delta \varepsilon_{jk}^p) d\Omega, \]  

If the incremental quantities are now separated from the initial ones, and taking into account the initial fields satisfy eq.(1), an incremental boundary integral equation is obtained:

\[ c_{ij} \Delta u_j + \int_{\Gamma} p_{ij}^* \Delta u_j d\Gamma = \int_{\Gamma} u_{ij}^* \Delta p_j d\Gamma + \]

\[ + \int_{\Omega} \sigma_{ijk}^* \Delta \varepsilon_{jk}^p d\Omega, \]  

The standard boundary element method procedure is applied next. After discretising the boundary \( \Gamma \), and the parts of the domain where \( \varepsilon^p \) are expected to be non-zero, eq (3) can be written in a matricial form:

\[ H \Delta u = G \Delta p + D \Delta \varepsilon^p, \]  

and in a similar fashion, the stress equation becomes now:

\[ \sigma = G' \Delta p - H' \Delta u + D^* \Delta \varepsilon^p. \]  

Rearranging equations (4) and (5) according to the boundary conditions,

\[ A \Delta y = \Delta f + D \Delta \varepsilon^p \]

\[ \Delta \sigma = -A' \Delta y + \Delta f' + D^* \Delta \varepsilon^p, \]  

The increments of plastic strains \( \Delta \varepsilon^p \) is a vector that appears in the standard boundary element analysis of elastoplastic problems, and is determined iteratively. Once convergence, is achieved, all incremental quantities are added to the accumulated values, and a new load step is allowed.
Contact

The application of the BEM equations (6) and (7) to the contact between two or more bodies, follows the same guidelines as for the conforming problems [7]. A potential contact area is chosen for each body, in such way that a number of nodepairs is defined. Outside the potential contact zone, usual boundary conditions, in the form of known displacements or tractions must be prescribed. Equation (6) is then written for each body. Neither displacements nor tractions are known inside the contact area, and therefore they are kept in the vector $\Delta \mathbf{y}$. The compatibility and equilibrium conditions are enforced within the contact zone, providing the additional equations needed to solve the problem.

The contact modes at each nodepair can be either stick, slip or separation, and are usually presented to express the total equilibrium and compatibility conditions. These must be always verified, but it is necessary to rewrite them in terms of the incremental quantities, to agree with the loading procedure. A local coordinate system $t, n$ is used, in which $n$ has the direction of the average normal to the boundaries of the bodies (see ref. [4]). For stick mode, it is possible to write:

\[
(p_i^a + \Delta p_i^a) + (p_i^b + \Delta p_i^b) = 0
\]

\[
(p_n^a + \Delta p_n^a) + (p_n^b + \Delta p_n^b) = 0
\]

\[
(u_i^a + \Delta u_i^a) - (u_i^b + \Delta u_i^b) = 0
\]

\[
(u_n^a + \Delta u_n^a) - (u_n^b + \Delta u_n^b) = gap^{ab},
\]

which can be further rearranged, in order to separate the accumulated quantities in the right hand side, (known from calculations for the previous load steps), from the unknown incremental quantities:

\[
\Delta p_i^a + \Delta p_i^b = -(p_i^a + p_i^b)
\]

\[
\Delta p_n^a + \Delta p_n^b = -(p_n^a + p_n^b)
\]

\[
\Delta u_i^a - \Delta u_i^b = -(u_i^a - u_i^b)
\]

\[
\Delta u_n^a - \Delta u_n^b = gap^{ab} - (u_n^a - u_n^b).
\]

Similarly, the contact constraints for the slip mode and for the separation mode are, respectively:

\[
\Delta p_i^a + \Delta p_i^b = -(p_i^a + p_i^b)
\]

\[
\Delta p_n^a + \Delta p_n^b = -(p_n^a + p_n^b)
\]

\[
\Delta p_i^a \pm \mu \Delta p_n^a = -(p_i^a \pm \mu p_n^a)
\]

\[
\Delta u_n^a - \Delta u_n^b = gap^{ab} - (u_n^a - u_n^b),
\]

\[
\Delta p_i^a + \Delta p_i^b = -(p_i^a + p_i^b)
\]

\[
\Delta p_n^a + \Delta p_n^b = -(p_n^a + p_n^b)
\]
\[ \Delta p_n^i = -(p_n^i) \]
\[ \Delta p_n^a = -(p_n^a) \]  

(11)

The description of the contact mode at any nodepair as "stick", "slip" or "separation" is to be understood only for the current load step. This conditions may change during the loading process, such that a stick region might have had a slip history, and vice versa.

**Determination of the load step**

The incremental solution of a given problem consists of closing the gaps of each nodepair, in succession. At each step, it is fundamental to ensure that the normal traction at the edge of the contact zone (i.e. at the nodepair being closed), fall to zero. If the normal traction is positive (tensile), it means that the applied load step is not big enough to bring the two nodes into contact. On the other hand, a negative (compressive) traction, means that the applied load is too large. The size of the load step has to be predicted.

It was proposed in [4], a load scaling technique, in which a trial load increment \( \Delta P_1 \) is applied. As a result, a value \( t_1 \) for the normal traction at the edge of the contact area is obtained. After this, a different load increment, \( \Delta P_2 \) is applied, and the corresponding value for the edge traction is now \( t_2 \). If the trial increments are small, and the material is linear elastic, a linear variation of the traction with the load can be assumed, and, thus, the correct load to apply, to guarantee zero edge traction is

\[ \Delta P_e = \Delta P_1 + \frac{t_1(\Delta P_2 - \Delta P_1)}{(t_1 - t_2)} \]  

(12)

However, if plasticity occurs during the current load step, the above assumptions are no longer valid, and the application of \( \Delta P_e \), will give a value of the edge traction \( t_3 \), which in general will not be zero. In the present paper, the load \( \Delta P_e \) is still used as a starting point for an iterative determination of the correct load increment to apply, \( \Delta P_{ep} \), which results in zero traction value at the edge.

**Incremental-iterative solution process**

When a new load step is applied, the contact mode of the nodepair next to the edge is set from separation to slip, becoming the new edge of the contact zone. After that, the load \( \Delta P_e \), is determined, by successive application of two trial loads, \( \Delta P_1 \) and \( \Delta P_2 \), and by making use of eq (12).

The contact modes are determined iteratively, that is to say, they are controlled, and if any violation is detected, they are changed. A violated slip mode is changed into stick, and a violated stick mode is changed into slip, taking care in this situation to set the frictional force to oppose the relative displacement.
Once the contact conditions are obtained, and the normal traction at the edge of the contact area is zero, the elastic solution for the current load step is known. The stresses at the internal nodes are checked against the yield stress, and if a plastic zone is detected, the iterative procedure to determine the increments of plastic strains, increments of boundary unknowns, and increments of internal stresses is carried out. The traction at the new edge of the contact zone (ie \( t_3 \)) is compared with zero, and if it is found to be positive, an increase of \( \Delta P_e \) is needed. On the contrary, if \( t_3 \) is negative, \( \Delta P_e \) has to be decreased. In this way, \( \Delta P_{ep} \) is obtained, when the normal traction vanishes. 

Once \( t_3 \) equals zero, the contact conditions, eqs. (9) (10) and (11), are checked. If any violation is detected they are corrected, ie, the matrix \( A^{-1} \) is updated, and the system of equations is solved again. After the computation of all the incremental quantities, the normal traction at the edge of the contact area is compared to zero, and the value of the load step is fine tuned, if necessary.

After convergence is achieved for the contact conditions and for the plastic algorithm, and the normal traction at the edge of the contact area is zero, the solution for the current increment has been obtained, and a new load step is allowed. The process ends when the maximum admissible area is reached, or the maximum load is applied.

**Numerical example: Cylinder on a flat foundation**

Consider the geometry shown in figure (1a), where the foundation has a semi-width \( W \) and a height \( H \). The ratio of the cylinder is \( R \). The ratios between them are assumed to be \( H/W = 2/1 \) and \( R/W = 5/9.5 \). Due to the symmetry of the problem, only half of the structure is modelled. The boundary mesh consists of 47 elements for the foundation, (of which 14 are situated in the potential contact area, ie linear interpolating functions are used within them), and 59 elements for the cylinder, 14 of which are linear.

Figure (1b) shows the domain discretisation, in the neighbourhood of the point \( A \), the only point where the two bodies touch each other in the undeformed state. For each body, 168 linear internal cells are used, covering an area \( w \times h \), which in the case of the foundation are \( w = 0.085W \), and \( h = 0.038H \).

A concentrated compressive load per unit thickness, \( P \), is applied vertically on the top of the cylinder.

Both cylinder and foundation are assumed to have the same material properties: elastic modulus \( E = 4000 \text{ Mpa} \); Poisson’s ratio \( \nu = 0.3 \); yield stress \( \sigma_y = 100 \text{ Mpa} \); plastic modulus \( H' = 595 \text{ Mpa} \) (work hardening material). The value of the coefficient of friction is \( \mu = 0.01 \). The elastic analysis is carried out for comparison purposes, assuming the same elastic properties, and elastic behaviour throughout the loading process.
The right hand side of figure (2) shows the normal and tangential tractions that develop in the contact area, as the load is applied incrementally. Up to the ninth load step, the behaviour of the structure is elastic, and while the load is being applied, a small slip area develops towards the edge of the contact zone. In the tenth load step, a plastic zone begins to appear, as the yield stress is reached at point A.

Further load steps show small increases of the normal traction in the centre of the contact area, because the plastified material is unable to increase its bearing capacity. In order to equilibrate the external load (which continues to raise), the normal tractions become larger at points close to the edges of the contact area, and then rapidly fall to zero, at the edge itself. The slip area continues to extend as the load becomes higher, such that only a small stick area remains near the symmetry axis.
Compare this with the fully elastic behaviour shown on the left hand side of figure (2). In this case, the normal traction at the central point raises steadily as the load is applied. It also reaches a maximum at that point. It is seen that a slip area increases its size as the successive load steps are applied, although it does not grow as wide as in the elastoplastic case.

Finally, figure (3) shows the distribution of the Von Mises stresses and equivalent plastic strains within the discretised domain, for the final value of the load. The plastic zone is slightly bigger in the cylinder, although it is similar to that observed in the foundation. It is interesting to notice a high gradient in the stress field, in the zone still under elastic regime, which roughly coincides with the edge of the contact area.

**Conclusions**

A boundary element formulation to solve non-conforming elastoplastic contact problems has been presented. This kind of problems are both non linear and dependent on the loading path, thus, an incremental loading technique has been developed.
The solution process consists of finding the value of the load step to apply, that is consistent with zero normal traction at the edge of the contact zone, together with the correct contact conditions and the increments of plastic strains. A direct constraint technique developed allows to couple the equations of the different bodies through equilibrium and compatibility conditions. These, in turn, depend on the contact modes, i.e., separation, stick or slip, which are obtained iteratively.

Elastoplasticity is solved by a BEM initial strain approach. The Von Mises yield criterion with its associated flow rule is adopted. Both perfectly plastic and work hardening materials can be handled in the proposed formulation. A numerical example was presented, where the results of an elastic analysis are compared to those of an elastoplastic one. The former are in good agreement with other published results.

References


