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Numerical solution of obstacle problems by overlapping block relaxation methods

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Abstract

Overlapping block relaxation methods are applied to the iterative solution of algebraic obstacle problems with M-matrices. Such problems arise, eg., from finite element approximation of obstacle problems with diffusion or convection-diffusion operators. We propose and analyze various algorithms with the emphasis on their monotonicity properties: the methods are monotonically convergent in the subset of supersolutions, and their asymptotic convergence rates can be evaluated in this particular subset. We also discuss the multilevel approach within the block relaxation methods; this approach makes possible to choose individual subproblem solvers for the methods. Numerical experiments are included to illustrate the theoretical results.

1 Introduction

In this paper, we investigate the numerical solution of obstacle problems which are closely related to various applications in contact mechanics. Especially, we develop iterative methods for algebraic obstacle problems with M-matrices, a class of problems arising from suitable mesh approximations of (interior or boundary) obstacle problems with diffusion and convection-diffusion operators. We have focused our attention on the development of the methods which can be treated within the unifying framework of overlapping block relaxation methods. We introduce and evaluate a methodology which can be efficiently used on serial and parallel computers.

Block relaxation methods (with nonoverlapping blocks) for variational inequalities have been considered in several publications, but mostly in the case of self-adjoint and coercive operators, when problems can be equivalent-

ly formulated as constrained minimization problems [3]. Also, the theory of nonoverlapping block relaxation methods for linear complementarity problems – problems closely related to obstacle problems – has been successfully developed [1]. On the other hand, it is known that Schwarz domain decomposition methods being applied to systems of mesh equations are particular cases of overlapping block relaxation methods; these methods have been applied to variational inequalities with self-adjoint and coercive operators in [4], [5], [9], [13], for instance. Extension of overlapping block relaxation methods to the case of nonself-adjoint operators has been developed in [6], [7], [8], [14]. In this paper, we review some theoretical and experimental results described in more detail in [7], [14].

The outline of the paper is as follows: Two variants of overlapping block relaxation methods (successive and simultaneous) are studied in Section 3.1. The methods are analyzed with the emphasis on their monotonicity properties: the methods are monotonically convergent in the subset of supersolutions, and we can also evaluate their asymptotic convergence rates in this particular subset. Special attention is given to the multilevel methods (Section 3.2) which make possible to choose individual (exact/inexact, linear/nonlinear) subproblem solvers within the block relaxation methods. Numerical results (Section 4) illustrate the effect of various subproblem solvers on the efficiency of the two-level block relaxation methods.

2 Problem formulations

Let Ω be an open bounded polygon in $\mathbf{R}^2$. We consider the following obstacle problem: find $u \in K_\psi$ such that

$$B(u, v-u) \geq (f, v-u) \quad \forall\, v \in K_\psi, \tag{1}$$

where $K_\psi \subset H_0^1(\Omega)$ is defined by: $K_\psi = \{v \in H_0^1(\Omega) |\ v \geq \psi \text{ a.e. in } \Omega\}$, $\psi \in H^2(\Omega)$ is an obstacle function such that $\psi|_{\partial\Omega} \leq 0$, and

$$(f, v) = \int_\Omega fv \, d\Omega, \quad f, v \in L^2(\Omega).$$

The bilinear form $B(\cdot, \cdot)$ corresponds to the convection-diffusion operator:

$$B(u, v) = \int_\Omega [a \, \nabla u \cdot \nabla v + (\mathbf{b} \cdot \nabla u)v] \, d\Omega, \quad u, v \in H_0^1(\Omega), \tag{2}$$

where the coefficients $a > 0$, $\mathbf{b} = (b_1, b_2)$ are piecewise smooth and bounded.

If $\mathbf{b} = (0, 0)$, then the problem (1) is equivalent to the minimization problem [3]: find $u \in K_\psi$ such that

$$J(u) = \min_{u \in K_\psi} J(v), \quad J : K_\psi \to \mathbf{R}, \quad J(v) = B(v, v) - 2(f, v). \tag{3}$$

Otherwise, the bilinear form (2) is nonsymmetric and not necessarily coercive because of the lower order terms. However, we assume here that the obstacle problem (1) has a unique solution [10]. By using discretization techniques like the finite element method with upwinding [12], we get the algebraic obstacle problem, which can be represented either in terms of variational inequalities: find $u \in K = \{v \in \mathbf{R}^n |\ v \geq \psi\}$ such that

$$(Au, v-u) \geq (f, v-u) \quad \forall\, v \in K, \tag{4}$$

or in the complementarity form: find $u \in S$ such that

$$(u-\psi)_j \cdot (Au-f)_j = 0, \quad \forall\, j \in \mathsf{N} \equiv \{1, \ldots, n\}, \tag{5}$$

where $f,\ \psi \in \mathbf{R}^n$. The subset S, defined by

$$S = \{v \in K |\ (Av-f)_j \geq 0,\ j \in \mathsf{N}\}, \tag{6}$$

is called the subset of supersolutions to the problem (4). We assume that the matrix $A \in \mathbf{R}^{n \times n}$ is an M-matrix [15]. Under the assumptions made, it can be shown that the algebraic problem (4) has a unique solution [1].

Remark 2.1 *Here, we consider the interior obstacle problem, but similar approach of overlapping block relaxation methods can also be applied to the numerical solution of boundary obstacle problems* [10]. *Overlapping domain decomposition methods for such problems have been studied in* [5], [13].

3 Overlapping block relaxation methods

3.1 Two variants of the block relaxation methods

Let the set $\mathsf{N}=\{1,\ldots,n\}$ of unknowns be partitioned into m overlapping groups $\mathsf{N}_1, \ldots, \mathsf{N}_m$. For given $w \in K$ and $1 \leq i \leq m$, we introduce the subset $K_i(w)$ of K by $K_i(w) = \{v \in K |\ v_j = w_j,\ j \notin \mathsf{N}_i\}$. Then, it is easy to show that the solution of the subproblem: find $v \in K_i(w)$ such that

$$(Av, \varphi - v) \geq (f, \varphi - v) \quad \forall\, \varphi \in K_i(w), \tag{7}$$

exists and is unique, and it can be represented in the form

$$v = T_i(w), \quad i = 1, \ldots, m, \tag{8}$$

where T_i is a nonlinear operator associated with the subproblem (7). In terms of the operators T_i, $i = 1, \ldots, m$, we next formulate two variants of the overlapping block relaxation methods.

Successive block relaxation method. Let $u^0 \in K$ be given. Then, according to the above partitioning of N, the successive block relaxation method is defined as follows:

$$u^{k+\frac{i}{m}} = T_i(u^{k+\frac{i-1}{m}}), \quad i = 1, \ldots, m,\ k = 0, 1, \ldots, \tag{9}$$

or, equivalently,

$$u^{k+1} = T(u^k) \equiv T_m(T_{m-1}(\cdots(T_1(u^k))\cdots)),\ k = 0, 1, \ldots. \tag{10}$$

Theorem 3.1 [7], [14] *The successive block relaxation method is monotonically convergent in the subset S of* (6), *that is, for any $u^0 \in S$, we have:*

$$u^k \in S, \quad u^k \leq u^{k-1},\ k = 1, 2, \ldots, \quad \text{and } \overset{*}{u} = \lim_{k\to\infty} u^k,$$

$\overset{*}{u}$ *being the solution of the algebraic obstacle problem* (4).

By using monotonicity arguments of the above method, we are able to evaluate its asymptotic convergence rate in the subset S: First, we introduce the matrices A_i related to the subproblems (7) for each $i = 1, \ldots, m$, i.e.,

$$A_i = P_i \begin{pmatrix} A_{11}^{(i)} & 0 \\ 0 & 0 \end{pmatrix} P_i^T \in \mathbf{R}^{n\times n}, \tag{11}$$

where, $A_{11}^{(i)}$ is the leading $n_i \times n_i$ diagonal block of the matrix $P_i^T A P_i$, $i = 1, \ldots, m$, and P_i is a suitable permutation matrix. It is known that every $A_{11}^{(i)}$ is also an M-matrix. We also introduce the matrices

$$H_i = A_i^+ = P_i \begin{pmatrix} [A_{11}^{(i)}]^{-1} & 0 \\ 0 & 0 \end{pmatrix} P_i^T, \quad i = 1, \ldots, m, \tag{12}$$

A_i^+ being the pseudo-inverse of the matrix A_i. Let R denote the iteration matrix of the above successive block relaxation method when it is applied to the linear system $Au_l = f$, that is,

$$R \equiv \prod_{i=1}^{m} R_i, \quad R_i = I - H_i A, \quad i = 1, \ldots, m. \tag{13}$$

Define a subset G^* of N (solution contact subset) by $\mathsf{G}^* = \{j \in \mathsf{N} |\ \overset{*}{u}_j = \psi_j\}$, and denote by A_* and $A_{*,i}$ the restrictions of the matrices A and A_i, $i = 1, \ldots, m$, respectively, to the subset $\mathsf{N} \setminus \mathsf{G}^*$ [14]. By using the above definitions, we can define the (linear) successive block relaxation method restricted to the subset $\mathsf{N} \setminus \mathsf{G}^*$ by ($R_{*,i} \equiv I_*$, if $\mathsf{N}_i \setminus \mathsf{G}^* = \emptyset$)

$$R_* = \prod_{i=1}^{m} R_{*,i}, \quad R_{*,i} = I_* - H_{*,i} A_*, \quad H_{*,i} \equiv A_{*,i}^+, \quad i = 1, \ldots, m. \tag{14}$$

Now, by defining the asymptotic rate of convergence of the iterative procedure (10) with respect to the subset $S \subset \mathbf{R}^n$ by [7]

$$\mathsf{R}_{S,\infty}(T) = -\lim_{k\to\infty} \left(\frac{1}{k} \ln \sup_{u^0 \in S} \frac{\| u^k - \overset{*}{u} \|}{\| u^0 - \overset{*}{u} \|} \right),$$

the following lower and upper estimates can be shown:

Theorem 3.2 [7], [14] *Under the assumptions made, we have*

$$-\ln \rho(R) \leq \mathsf{R}_{S,\infty}(T) \leq -\ln \rho(R_*), \tag{15}$$

$\rho(R)$ *denoting the spectral radius of the matrix* R.

Simultaneous block relaxation method. Assume again that the set N is divided into m overlapping subsets N_i, $i = 1, \ldots, m$. We define the operators T_i, $i = 1, \ldots, m$ by (8), and introduce positive numbers ω_i such that $\sum_{i=1}^{m} \omega_i = 1$. Then, the simultaneous block relaxation method can be represented in the form

$$u^{k+1} = \sum_{i=1}^{m} \omega_i T_i(u^k) \equiv T_\omega(u^k), \quad k = 0, 1, \ldots. \tag{16}$$

Theorem 3.3 [7], [14] *The simultaneous block relaxation method is monotonically convergent in the subset* S, *and the following estimations hold:*

$$-\ln \rho(R_\omega) \leq \mathsf{R}_{S,\infty}(T_\omega) \leq -\ln \rho(R_{*,\omega}), \tag{17}$$

where $R_\omega = \sum_{i=1}^{m} \omega_i R_i$ *and* $R_{*,\omega} = \sum_{i=1}^{m} \omega_i R_{*,i}$, R_i *and* $R_{*,i}$, $i = 1, \ldots, m$, *being defined by* (13) *and* (14), *respectively.*

3.2 Multilevel methods

Here, for the sake of simplicity, we only describe the simplest variant of multilevel approach, namely two-level block relaxation methods. Generalization to other multilevel methods is straightforward.

Let us consider the algebraic obstacle problem (4), where A is assumed to be an M-matrix. Assume further that the set N of unknowns is partitioned into m overlapping subsets N_i. This partitioning into subgroups defines the first level, the outer level, of the two-level block relaxation methods.

By analogy to Section 3.1, we introduce the iteration operators T_i, $i = 1, \ldots, m$, associated with the subproblems defined by (7). After that, we introduce new operators $\mathcal{T}_i$, $i = 1, \ldots, m$, with the following assumptions:

(i) The subset S is invariant with respect to the operators $\mathcal{T}_i$, $i = 1, \ldots, m$;

(ii) For all $w \in S$, we have $\mathcal{T}_i(w) \leq w$, and

$$\mathcal{T}_i(w) < w \iff T_i(w) < w \quad \text{for any } w \in S,\ w \neq \overset{*}{u};$$

(iii) Operators $\mathcal{T}_i$, $i = 1, \ldots, m$, are continuous in S.

The operators $\mathcal{T}_i$, $i = 1, \ldots, m$, define the inner level of two-level methods.

Two-level successive block relaxation method. Following the presentation of Section 3.1, we define the two-level successive block relaxation method via the operators $\mathcal{T}_i$, $i = 1, \ldots, m$, by

$$u^{k+1} = \mathcal{T}(u^k) \equiv \mathcal{T}_m(\mathcal{T}_{m-1}(\cdots(\mathcal{T}_1(u^k))\cdots)), \ k = 0, 1, \ldots. \tag{18}$$

Theorem 3.4 [14] *The two-level successive block relaxation method is monotonically convergent to the solution* $\overset{*}{u}$ *of* (4) *in the subset* S.

Two-level simultaneous block relaxation method. We next formulate the two-level simultaneous block relaxation method via the operators $\mathcal{T}_i$, $i = 1, \ldots, m$ (following the notations of Section 3.1) by

$$u^{k+1} = \sum_{i=1}^{m} \omega_i \mathcal{T}_i(u^k) \equiv \mathcal{T}_\omega(u^k), \ k = 0, 1, \ldots. \tag{19}$$

Theorem 3.5 [14] *The two-level simultaneous block relaxation method is monotonically convergent to the solution* $\overset{*}{u}$ *of* (4) *in the subset* S.

The advantage of the proposed approach is that for any subproblem, we can choose an individual operator $\mathcal{T}_i$ to solve the problem. For instance, we can replace exact subsolvers by various inexact ones. On the other hand, it provides the possibility to partly use linear subsolvers for the genuinely nonlinear obstacle problems, provided we can find a suitable block partitioning of the matrix A. One interesting technique to obtain such information is the technique of two-sided approximations, described in [8], [14].

4 Numerical results

The aim of the numerical experiments is to illustrate the effect of the two-level approach on the efficiency of the overlapping block relaxation methods. More numerical results related to these methods can be found in [14].

We study numerically whether it is possible to accelerate the solution of algebraic obstacle problems by introducing various subsolvers (exact/inexact, linear/nonlinear) within the two-level successive block relaxation methods. Let us consider the two-dimensional obstacle problem (1) with the following assumptions: $\Omega = (0,1) \times (0,1)$, $a = 1$, the convection term $\mathbf{b} = (5,5)$, the obstacle function $\psi \equiv 0$, and the function f is defined by

$$f(x) = \begin{cases} -2, & x \in (3/8, 5/8) \times (3/8, 5/8) \\ 1, & x \notin (3/8, 5/8) \times (3/8, 5/8). \end{cases}$$

We apply the finite element method with upwinding on the uniform mesh ($h = \frac{1}{128}$) to discretize the problem. We define the outer level of the two-level

successive block relaxation method via the following overlapping domain decomposition: we decompose the domain Ω into three rectangular strips Ω_1, Ω_2, and Ω_3 in such a way that the solution contact subdomain is included in $\Omega \setminus (\Omega_1 \cup \Omega_3)$. Hence, within the block relaxation methods in the subset S, we can solve linear subproblems in subdomains Ω_1 and Ω_3.

The inner level of two-level methods is defined as follows: In Ω_2, we always apply p inner iterations of the line-Gauss-Seidel method such that the tridiagonal subproblems are solved by Cryer's method [2]. Instead, for the linear subproblems in Ω_1 and Ω_3, we test three different subsolvers: (nonlinear) line-Gauss-Seidel with Cryer's method as the tridiagonal solver (A1 in the tables below), line-Gauss-Seidel with Gaussian elimination (A2), and a fast linear solver based on FFT and symmetrization (A3) [11].

In Tables 1 and 2, we have reported the required numbers of iteration steps and the total execution times (on HP 9000/735 computer), respectively. First, we see that by increasing the numbers of inner iteration steps, or, by solving the subproblems with high accuracy ("exactly"), we can remarkably decrease the numbers of outer iteration steps. However, the total execution time is not necessarily decreased. In fact, we can deduce from these experiments that it is not reasonable to execute very many inner iteration steps within each (outer) iteration step. In this sense, the approach of two-level methods with inexact subproblem solvers seems to be useful from the point of view of practical implementation. Also, we notice that by replacing nonlinear subsolvers by faster linear ones in suitable subdomains (algorithms A2 and A3), it is possible to obtain notable reduction of the corresponding execution times. This confirms another advantage of the two-level approach within the block relaxation methods, if we have information about the linear/nonlinear decomposition of obstacle problems.

p	A1	A2	A3
1	1511	1511	1294
2	849	849	722
5	393	393	332
10	221	221	186
"exact"	73	73	65

Table 1: Numbers of iterations.

p	A1	A2	A3
1	40.56	36.36	53.10
2	**38.64**	**32.53**	37.24
5	38.71	33.20	27.15
10	44.37	34.78	**24.30**
"exact"	67.93	56.85	34.75

Table 2: Execution times (CPU seconds).

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