Unsymmetrical indentation of an elastic half-plane in the presence of friction reversal

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ABSTRACT

Static rigid indentation of a linearly elastic half-plane in the presence of Coulomb friction that reverses its direction along the contact length is studied in this paper. A rigid punch, having an unsymmetrical profile in respect to its apex and no concave regions, slowly indents an elastic body even as it slides over it. Both a normal and a tangential force may be exerted on the indentor, therefore. In such a situation, depending upon the indentor shape and the relative magnitude of the external forces, a point at which the surface friction changes direction within the contact region, may exist. Moreover, this point of sign reversal in the shear traction along the half-plane surface does not coincide, in general, with the indentor apex; that position being unspecified at the outset and determined only by solving the problem. The latter fact complicates the problem and requires the numerical solution of a system of non-linear equations containing integrals not available in closed form.

INTRODUCTION

In recent years, progress in Contact Mechanics has often been associated with going beyond the restriction of frictionless surfaces (see e.g.[1-7]). For elastostatic problems particularly, the 2-D frictional indentation of a half-plane by a rigid curved punch which both indents the deformable body and slides over its surface was considered by Muskheilishvili [8] and Erdogan [9], among others. In this problem, Coulomb's friction acts only in one direction on both sides of the indentor. The more involved problem of a symmetrical curved or pointed punch indenting a half-plane under normal loads, where Coulomb's friction acts in opposing directions on opposite sides of the
indentor, was considered by Roberts [10], and by Brock and Georgiadis [11].

In the present work, we consider a generalization of the latter problem, i.e. the elastostatic frictional indentation of a half-plane by an unsymmetrical punch under normal and tangential force (see Fig. 1). Depending upon the indentor's shape and the relative magnitude of the external forces, a point at which the surface friction changes direction, within the contact region, may exist. Thus, the present problem can also be considered as a generalization of the classical frictional contact problem, studied in Refs. [8,9]. In the present case, the length defining the position of sign reversal of the shear traction along the half-plane surface, \((x=-c,y=0)\), and the lengths \(a\) and \(b\) defining the contact zone are three unknowns to be determined by first obtaining the contact-stress expressions (in terms of the unspecified quantities) and then enforcing pertinent equilibrium and consistency equations.

The mathematical procedure consists of solving the Navier-Cauchy partial differential equations supplied with boundary conditions of a mixed type. We succeeded in formulating a Cauchy singular integral equation of the second kind and solving it exactly by the Riemann-Hilbert analytic function theory. Although a general indentation profile is considered, explicit results are given only for the wedge- and parabolic-shaped indentor cases. In particular, examining the unsymmetrical wedge problem allows for uncoupling the effect of friction reversal from that of the geometrical corner-discontinuity. In the respective symmetrical problem [11], both these effects correspond to the same point, i.e. the wedge apex, and thus it is interesting to study separately each one.

Our analysis shows that a power-type singularity in contact stress occurs about the point of shear sign-reversal. Such a stress singularity may well produce cracking in the solid beneath the punch, and thus the present results are of certain value to relative investigations in Penetration and Fracture Mechanics [12]. Finally, it is noted that in our modeling of the problem, no adhesion (stick) zone is assumed to develop along the contact zone, leading to a formulation with purely mixed boundary conditions. Therefore, the present formulation is adequate only for situations such as non-flat indentors and weak friction.

**PROBLEM STATEMENT**

Consider an isotropic linearly elastic half-plane \(y \leq 0\) having Lame's constants, \(\lambda\) and \(\mu\), under a rigid indentation over a contact zone \((a+b)\), see Fig. 1. The origin of the coordinate system is placed in the indentor's apex (lowest point of the punch profile). The boundary conditions for an unsymmetrical pointed or curved punch profile with no concave regions, \(G(x)\), in the presence of friction reversal, are then written as

\[
\begin{align*}
\sigma_y &= G(x) \quad \text{for } -b<x<a, \quad y=0 \quad (1.1) \\
\sigma_{xy} + \text{sgn}(x+c) \gamma \sigma_{yy} &= 0 \quad \text{for } -b<x<a, \quad y=0 \quad (1.2)
\end{align*}
\]
\[ \sigma_{yy}, \sigma_{xy} = 0 \quad \text{for } -\infty < x < -a \text{ and } a < x < \infty, \quad y = 0 \quad (1.3) \]

where \( \text{sgn}(\ ) \) is the signum function, \( \gamma \) is the friction coefficient (positive constant), and the convention is followed whereby \( \sigma_{xy}(x, y=0) > 0 \) when pointing the positive \( x \)-axis.

![Figure 1: Static rigid indentation of an elastic half-plane in the presence of friction reversal.](image)

Under Equations (1) and the pertinent Navier-Cauchy elastostatic equations, we intend to determine the contact-stress distribution \( \sigma_{yy}(-b < x < a, y=0) \), and the unknown lengths \( a, b \) and \( c \), in terms of the indentor-profile geometry \( G(x) \), and the external forces applied to the indentor, \( N \) and \( S \).

**ANALYSIS**

The above problem will be solved by a singular integral equation approach. The basic result, from plane elasticity theory, which is utilized here is the following relation connecting \( \frac{\partial u_y}{\partial x} \), \( \sigma_{yy} \) and \( \sigma_{xy} \) between each other along \( -\infty < x < \infty, \quad y = 0 \) (see e.g. [9], Sect. 7)

\[
\frac{4\mu}{1+v^*} \frac{\partial u_y(x,0)}{\partial x} = -\beta \sigma_{xy}(x,0) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma_{yy}(\tau,0)}{\tau - x} \, d\tau \quad (2)
\]

where \( v^* = 3-4v \) for plane strain, \( v^* = (3-v)/(1+v) \) for plane stress, \( \nu = \lambda/2(\lambda+\mu) \) is Poisson's ratio and \( \beta = (v^*-1)/(v^*+1) \) is a new material constant (positive). We also introduce the contact pressure function, \( p(x) = -\sigma_{yy}(x,0) \) for \(-b < x < a\), and the tangential derivative of the indentor profile, \( g(x) = \frac{dG(x)}{dx} \).

By now combining Equations (1) with (2), and employing the above definitions, the integral equation for the problem
\[ \text{sgn}(x+c) \omega p(x) - \frac{1}{\pi} \int_{-b}^{a} \frac{p(t)}{t-x} \, dt = f(x) \quad (3) \]

is obtained, where \( \omega = -\beta \gamma, \omega < 0, \) and \( f(x) = 4\mu g(x)/(1+\nu^2) \). The solution to this integral equation can be effected by following Carleman's technique \([8,9]\). We omit the details of this procedure and refer to \([13]\) for a full derivation. We notice however that a Riemann–Hilbert problem is formulated; the latter having a fundamental function (partial solution) exhibiting Holder discontinuity on the point \((x=-c, \gamma=0)\).

Finally, by supplying the mathematical solution with the Signorini contact conditions (i.e. \( \sigma_{yy}(x,0) \leq 0 \) for \(-b < x < a\), and \( u_y(x,0) \leq G(x) \) for \(-\infty < x < -b \) and \( a < x < \infty \)) and some arguments regarding integrable singularities, the solution of the problem can be obtained in the form

\[
\sigma_{yy}(x,0) = - \frac{|x+c|^2 \psi^{-1}[(a-x)(b+x)]^{1-\psi}}{(\omega^2 + 1)n}.
\]

accompanied with the so-called consistency equation \([9,13]\)

\[
\int_{-b}^{a} \frac{f(t)}{(t-x)|t+c|^2 \psi^{-1}[(a-t)(b+t)]^{1-\psi}} \, dt - \frac{\text{sgn}(x+c) \omega}{\omega^2 + 1} f(x)
\]

for \(-b < x < a \quad (4)\]

where \( \psi = -(1/m) \tan^{-1}(1/\omega) \), with \( 0 \leq \psi \leq 1/2 \) and \( \psi = 1/2 \) for \( \gamma = 0 \) (frictionless limit).

It is seen from (4) that the contact stress depends upon the as yet unknown lengths \( a, b \) and \( c \). However, these lengths can be determined by enforcing Equation (5) along with the two equilibrium conditions (involving integrations of \( \sigma_{yy}(x,0) \) and \( \sigma_{xy}(x,0) \) over \(-b < x < a\)) which relate the normal and tangential contact stress with the respective external forces applied to the indentor. In this way, a system of three non-linear equations is formed, which, when solved numerically by iteration, may provide the values of the contact lengths \( a, b \) and \( c \). Then, the contact stresses are computed by (4) and (1.2).

We notice, however, that the integral in (4) is available
in closed form only for a few special cases of indentation profile (the unsymmetrical continuous-parabolas indentation considered in the next Section is one of them). In general, one has to perform numerical integration for the Cauchy-type integral in (4) (as is done for the unsymmetrical wedge indentation considered in the next Section), which is not too much of a problem, but may complicate the subsequent numerical procedure of solving the pertinent non-linear equations.

RESULTS AND CONCLUDING REMARKS

Representative numerical results have been obtained for two common cases of indentation profiles, namely the parabolic- and wedge-shaped indentors.

Let us first consider the unsymmetrical continuous-parabolas case defined by the indentation profile

\[ G(x) = \begin{cases} \frac{x^2}{2R} - D & \text{for } x > 0 \\ \frac{x^2}{2Q} - D & \text{for } x < 0 \end{cases} \]  

(6.1) \hspace{1cm} (6.2)

where D, Q and R are positive constants. In this case, \( f(t) \sim t \), and accordingly the integrand in (4) takes a form which permits a closed-form evaluation by using contour integration and certain properties of Cauchy integrals. A straightforward evaluation in (4) gives the normal contact stress for the unsymmetrical parabolic indentation as

\[ \sigma_{yy}(x,0) = -\frac{H}{Q} |x+c|^\frac{2\psi-1}{1-\psi} \left[ (a-x)(b+x) \right]^{\frac{1-\psi}{1-\psi}} \]  

for \(-b < x < 0\) (7.1)

\[ \sigma_{yy}(x,0) = -\frac{H}{R} |x+c|^\frac{2\psi-1}{1-\psi} \left[ (a-x)(b+x) \right]^{\frac{1-\psi}{1-\psi}} \]  

for \(0 < x < a\) (7.2)

where

\[ H = 4\mu / [(1 + \nu^*) (\omega^2 + 1)^{1/2}] \].

Next, we obtain numerical values of the contact lengths for specific values of parameters related to geometry (radii Q and R), material (shear modulus \( \mu \), Poisson's ratio \( \nu \), and friction coefficient \( \gamma \)) and loading (external forces \( N \) and \( S \)). In solving the non-linear system of equations, we have employed a routine of the standard computer program MATHCAD which is based on an iterative scheme. In this way, by setting \( R = 0.1m \), \( \mu = 1.2 \times 10^6 \) (Newton/m^2), \( \nu = 0.3 \), \( \nu^* = 2.0769 \), \( \gamma = 0.2 \), \( \omega = 0.4777 \), \( N = 1.2 \times 10^6 \) (Newton/m), the results in Table 1 are obtained for variable Q and S.

<table>
<thead>
<tr>
<th>Q = 0.50R</th>
<th>b</th>
<th>c</th>
</tr>
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<tbody>
<tr>
<td>S = 0.20N</td>
<td>0.0069</td>
<td>0.0048</td>
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Table 1:
The latter results permit some insight into the problem. Firstly, with an increasing tangential external force $S$ the lengths $b$ and $c$ are increased, for constant radius $Q$. Especially the length $c$ is radically changed and tends fast to become greater than $b$, leading thus to a situation with no reversing friction in which case our solution degenerates to the one obtained by Muskhelishvili ([8], Sect. 100). Secondly, with an increasing radius $Q$ and keeping $S$ constant now, the lengths $a$ and $c$ decrease, whereas the length $b$ increases. This means that making the indentor's profile more and more symmetrical the contact length becomes more and more symmetrical about the indentor's apex, whereas the position of friction sign-reversal moves to the central position (apex). The foregoing results may be expected intuitively but, of course, this analysis permits for quantitative clues.

We now direct attention to the unsymmetrical wedge indentation defined by the punch profile

$$G(x) = \begin{cases} \cot(\theta_+) x & \text{for } x > 0 \\ -\cot(\theta_-) x & \text{for } x < 0 \end{cases}$$

(8.1) \hspace{1cm} (8.2)

where $\theta_+$ and $\theta_-$ are the inclination angles of the wedge. In this case, $f(t)^{(\text{const.}^*)}$, the integral in (4) can no longer be evaluated in closed-form, and we have to resort to numerics. Attention should be paid because of the Cauchy-type kernel and end-point singularities (which are integrable, though) in the integrand. Without entering into further details, a graph for the contact pressure distribution, is presented in Figure 2. Specifically, the variation of the normalized contact stress

$$\frac{[-(1+\nu^*)(\omega^2+1)/4\mu]}{\sigma_{yy}(x,0)}$$

along a normalized contact length $(-b,a)=(-0.8a,a)$ is given, for $c=0.3a$, $\theta_+=87^\circ$, $\theta_- = 84^\circ$, $\mu=1.2 \times 10^9$ (Newtons/m$^2$), $\nu=0.3$, $\nu^*=2.0769$, $\gamma=0.2$ and $\varphi=0.4777$. One can observe the very steep distribution around the strongest algebraic singularity at $x=-c$, in respect to the relatively smoother dis-

<table>
<thead>
<tr>
<th>$Q$</th>
<th>$S$</th>
<th>$0.0069$</th>
<th>$0.0047$</th>
<th>$0.0031$</th>
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<tr>
<td>$0.25R$</td>
<td>$0.00N$</td>
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<td>$0.0058$</td>
<td>$1.43 \times 10^{-4}$</td>
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</table>
tribution around the weaker logarithmic singularity at $x=0$. Finally, notice that both these singularities are predicted by a qualitative analysis, in view of the contact stress expression in (4) and the particular indentation profile in (8).

![Contact stress (normal) distribution for an unsymmetrical wedge indentation in the presence of friction reversal.](image)

Figure 2: Contact stress (normal) distribution for an unsymmetrical wedge indentation in the presence of friction reversal.

REFERENCES

6. Hayashi, K. and Abe, H. "The Elastic Field near a Point of a


