Analysis of two-dimensional and axisymmetric contact and friction problems using boundary element software
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ABSTRACT

This paper presents the analysis of frictional contact problems using the boundary element software package, BEACON, in which three alternative approaches of enforcing the contact constraints have been developed and implemented. The first approach requires node-on-node matching along the interface while the other two allow free user-meshing of the interacting surfaces. In addition, the program incorporates several contact interface features such as frictional slipping, heat conduction, thermoelastic and clearance/interference fitting. Several examples are presented in which the results from the three approaches are shown to be in very good agreement with each other and with other available solutions.

INTRODUCTION

The analysis of contact problems with friction is an important part of most practical engineering applications, such as in pressure vessels, ball- and roller-bearings and hip-joints. These problems are highly non-linear because of two main reasons: first, the extent of the contact zone between the contacting bodies is usually unknown in advance and has to be determined by making successive guesses which are refined during the solution procedure, and second, the presence of friction between the contacting surfaces introduces a stick-slip partitioning of the contact zone, the extent of which is also unknown \textit{a priori} and is dependent on the magnitude of the coefficient of friction. These features limit the applicability of the available analytical solutions (see, for example, Johnson [1]) to simplified geometries and loadings thereby making the development of numerical approaches imperative. Numerous solution approaches via the finite element (FE) method have been developed, an excellent survey of which has been presented by Pascoe \textit{et al.} [2]. These FE approaches fall into three main categories namely, the penalty function, flexibility matrix and Lagrange multiplier methods. The first two require node-on-node contact within the interface, while the third introduces
independent Lagrange multipliers as additional variables into the solution matrix to allow mismatching of the contacting nodes. The efficiency of the FE approaches is, however, limited due to their domain discretisation approach and the displacement-based nature of the equations, the latter making it difficult to simultaneously satisfy the compatibility and equilibrium conditions within the contact interface.

The boundary element (BE) method is particularly well suited to contact problems, due to its boundary-only modelling and the relatively high accuracy of the stresses, when compared to the FE method. Thus the BE method leads to a substantial reduction in the size of the problem and, since the tractions and displacements are contained as field variables and are computed to the same degree of accuracy, the contact constraints can be exactly imposed in the system of equations. This paper presents the analysis of contact problems with friction using the BE software, BEACON, which incorporates three alternative approaches of enforcing the contact constraints in the system of equations. In the node-on-node approach (see, for example, Andersson et al. [3], Karami [4] and Abdul-Mihsein et al. [5]) the constraints are applied at discrete nodal points from either domain which coincide with each other when in contact, while the mapped-element approach (Olukoko et al. [6]) employs the element shape functions as interpolation polynomials in enforcing the constraints between each individual contact node and its opposite contact segment in the other domain, thereby permitting independent discretisation of the contacting surfaces. The fictitious-node approach (Olukoko et al. [7]) is also based on mismatching of nodal points along the interface and uses fictitious nodes to impose the contact constraints, thus resulting in the added advantage of allowing a fine mesh discretisation on one contact surface and a relatively coarser mesh on the opposite surface. These approaches are developed for static and proportional loading cases only, with linearly elastic, homogeneous and isotropic material behaviour. The software aspects and applications of BEACON are the main objectives of this paper. Examples are presented to demonstrate the accuracy of the three approaches.

THEORY

The determination of the extent of the contact zone and the displacements and stresses within the interface is the primary aim of numerical approaches in contact analysis. In the light of this, contact problems may be classified into two main categories namely, frictionless and frictional. The former is closely approximated by a fully-lubricated interface, in which case only normal compressive stresses are present at the contact surface and relative tangential deformation of the contacting nodes is allowed. For frictional contacts, however, the interface permits normal compressive and shear stresses and may be partitioned into regions of stick and slip depending on the magnitude of the coefficient of friction, \( \mu \). The former arises when the ratio of the tangential to the normal traction at a node is lower than the magnitude of \( \mu \), while slip
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occurs if this ratio is greater than or equal to $\mu$. The solution of contact problems requires the enforcement of certain boundary conditions within the common interface of the contacting bodies in accordance with the current contact modes of frictionless and frictional. For frictional sticking, equilibrium of tractions and continuity of displacements are imposed in both the normal and tangential directions thereby impeding relative deformation within the interface. Relative tangential deformation is allowed in the tangential direction for frictional slipping, hence continuity of displacements is applied in the normal direction only while equilibrium of tractions is imposed in the normal and tangential directions. The local tangential traction is also set equal to the product of the local normal traction and the coefficient of friction according to Coulomb’s law of limiting friction. The slip conditions are equally applicable to frictionless contact with $\mu$ set equal to zero. Details of the implementation of these constraints for the three contact modelling approaches can be found in Olukoko et al. [6,7,8].

THE BOUNDARY ELEMENT SOFTWARE, BEACON

BEACON is a multi-domain two-dimensional and axisymmetric BE program for continuum mechanics problems written in standard Fortran. It has an in-built mesh generator that can be used to produce a BE mesh of the domains to be analysed, which will usually consist of a series of circular and straight line segments. Minimum input is required from the user in this respect in the form of the coordinates of the keypoints at the beginning and end of each segment, and the number of elements and the variation of the element lengths along the segments. The program employs three-noded isoparametric elements having a quadratic variation of the geometry and unknown variables on each element. Boundary conditions are accepted on each element (or an entire segment) in the form of prescribed tractions, stresses or displacements. A comprehensive error-checking routine is included to detect trivial and fatal errors in the input data file, such as an element with over-prescribed boundary conditions. The program is based on a direct formulation of the BE method, details of which are contained in the book of Becker [9].

Contact Modelling Options

The user is allowed a free choice between the node-on-node, mapped-element and fictitious-node approaches. BEACON also offers options of the solution method where the user can specify automatic iterations for the contact modes or glued interface conditions. For the former, details of which will be discussed later, an upper limit may also be specified for the number of iterations. Glued interfaces may result from the presence of an external bond or adhesive between the contacting surfaces, in which case no iterations are performed and the analysis is terminated at the end of the first solution stage. The contact capabilities of the program also include thermoelastic (additional effects of temperature variations), heat source (presence of a source of heat at the interface), clearance/interference fitting and spring stiffness between the
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contacting surfaces.

Figure 1: Discretisation of the Contact Surfaces

**Node-On-Node Approach** This method requires the discretisation of the pair of contacting surfaces into the same number of elements to form element-pairs, the two opposite elements in a pair having equal lengths prior to deformation and the opposite nodes on each element-pair designated as nodepairs, as illustrated in Figure 1. The user inputs the two elements in a pair, the contact code and the coefficient of friction. Thus, \( n \) lines of data would be required for \( n \) element-pairs in the contact zone. BEACON then enforces the contact constraints between the nodepairs on each element-pair which coincide with each other when in contact (see Olukoko et al. [8]).

**Mapped-Element Approach** The contacting surfaces may be discretised independently into elements and nodes without the need for node-on-node matching as shown in Figure 1. For each pair of contacting surfaces, user-input is required in the form of the keypoints defining the beginning and end of each curve, the contact code and the coefficient of friction. Hence only one line of data is required for every pair of contacting surfaces irrespective of the number of elements on these curves. BEACON has an automatic length mapping technique that searches for all the nodes on each contacting curve and subsequently determines the relative location of each of these contact nodes on the opposite contact segment in the form of its opposite contact element and its intrinsic coordinate on this element - a typical mapped location point is indicated by the letter \( X \) on surface A in Figure 1. The contact constraints are then enforced between each contacting node and its location point by using the quadratic element shape functions to distribute the geometry and contact variables on the opposite contact element, as detailed in Olukoko et al. [6].

**Fictitious-Node Approach** User-input for this approach is exactly the same as for the mapped-element since both are based on independent discretisation of the contacting surfaces. BEACON determines the finer of the pair of contacting surfaces from their relative mesh discretisations and, by employing the same length mapping technique as for the mapped-element method, computes the mapped location point of each contact node on the finer curve.
relative to the coarser opposite curve. The program then uses the shape functions to distribute the geometry on the opposite contact element and introduces a fictitious node at the mapped location point, the existence of which is not apparent to the user, in order to enforce the contact constraints as described by Olukoko et al. [7]. Thus for the potential contact surfaces in Figure 1, where surface B has a relatively finer discretisation, a mirror-image of the nodes and elements on this surface is formed onto the opposite surface A. A typical fictitious node is therefore represented by the point X on surface A in Figure 1. Consequently, for a pair of contacting surfaces, the fictitious-node approach facilitates the employment of a relatively coarser mesh discretisation on one of curves and a finer one on the other, and gives numerical results representative of employing the finer mesh on both contact surfaces.

**Solution Method**

The boundary integral equation formulation results in a fully-populated, banded and unsymmetric solution matrix for each body in contact. After applying the prescribed boundary conditions, the matrices for the bodies in contact are coupled together according to the chosen contact approach and the current contact conditions. BEACON solves the final system of equations using a Gaussian elimination solver. Due to the inherently nonlinear nature of contact problems, an automatic iterative procedure without load incrementation is incorporated in BEACON to determine the final extent of the contact zone and the stick-slip partitioning of the contact area - this technique has been demonstrated by the authors (see Olukoko et al. [6,7]) to give reliable results of the contact variables and the extent of the contact area for proportional loading cases, since the stress field is similar at each stage of the loading. At the beginning of each analysis, the user makes an initial guess of the contact area which serves as a starting-point for the program. Three main checks are performed at the end of each solution stage for overlap/interpenetration, stick-slip partitioning and tensile stress detection, and the new contact conditions are imposed on the relevant nodes in the next solution. For the overlap check, the nodes lying in the periphery of the contact area are checked for interpenetration into the opposite domain, the occurrence of which indicates an initially small estimate of the contact area. The overlapping nodes are therefore included as part of the contact zone in the subsequent solution stage. For frictional contact problems all the contacting nodes are initially assigned a sticking status. If the ratio of the calculated values of the tangential to the normal traction at a node exceeds the magnitude of the coefficient of friction, μ, then a slip status is assigned to this node in the subsequent solution. Tensile stresses will appear within the contact zone if the initial estimate of the contact area is too large for the applied load. Hence, contact nodes that develop tensile normal stresses are excluded from the contact zone in the subsequent iteration. Convergence of the solution process is attained when no errors are detected by the aforementioned checks.
EXAMPLES

A Rigid Roller Between two Elastic Layers
The thermoelastic contact of a rigid cylindrical roller trapped between two relatively hot elastic layers is considered, in which the elastic layers expand under increasing temperature and make contact with the roller. Only a quarter of the roller-surface arrangement is modelled due to the symmetry about the horizontal and vertical axes, and a linear temperature distribution in the surface layers is considered. The following physical dimensions and material properties are employed under plane strain assumptions; roller radius, \( R = 1.0 \), modulus of elasticity, \( E = 0.6875 \times 10^{11} \), Poisson ratio, \( \nu = 0.3 \), coefficient of expansion, \( \alpha = 0.3 \times 10^{4} \), coefficient of friction, \( \mu = 0 \) or 0.3, temperature change per unit depth, \( T = 100 \). The BE mesh shown in Figure 2 consists of 64 isoparametric quadratic elements for the node-on-node approach with 10 element-pairs at the expected contact interface, while for the mapped-element and fictitious-node approaches, the expected contact zone consists of 8 and 6 elements respectively on the lower surface of the rigid roller, while the remainder of the model is as for the node-on-node.

![BE Mesh](image)

**Figure 2 : A Rigid Roller Between Two Elastic Layers**

The normal contact stress distributions for \( \mu = 0 \) and 0.3 are shown in Figure 3(a) together with the FE results and the Hertzian analytical solution for \( \mu = 0 \) \((P_0 = 1.1135 \times 10^{9} \) is the maximum contact pressure for the Hertzian case). It can be observed that, relative to the frictionless results, the contact width decreases by about 2% while the maximum contact pressure increases by about 8% for \( \mu = 0.3 \). The shear stress results obtained from the three BE approaches for \( \mu = 0.3 \) are shown in Figure 3(b) together with the FE solutions. Approximately half of the contact area is obtained as slipping from the BE approaches, while the FE results give a slip zone of about 45%. This accounts for the slight gap between the peak of the shear stress distributions from the BE and FE results in Figure 3(b).
An Elastic Plate on a Rigid Base

The contact problem between a semi-infinite elastic plate and a rigid base is considered. The elastic plate is subjected to a uniform pressure of magnitude $P_0$ on its top surface and a concentrated load of magnitude $P = rP_0h$ (where $h$ is the thickness of the elastic plate and $r$ a scale factor) at the centre as shown in Figure 4. Plane strain conditions are assumed and the following data are used in the analysis: modulus of elasticity of plate, $E = 1.0$, $v = 0.3$, $\mu = 0.3$, $P_o = 1.0$, aspect ratio of symmetric half considered, $h/b = 1/20$. Figure 4 also shows the BE mesh for the node-on-node approach consisting of 43 and 42 elements respectively for the elastic plate and rigid base, in which the contact surface is made up of 31 element-pairs. For the other two approaches, the BE mesh for the elastic plate remains unchanged while the rigid base is modelled with 31 and 8 elements (20 and 2 on the contact surface), respectively, for the mapped-element and fictitious-node approaches. Thus, for a large contact area example such as the present one, using the fictitious-node approach leads to a significant reduction in the number of elements used to model one of the two contacting surfaces and the domain containing it.

Figure 4: An elastic plate on a rigid base
Figure 5 shows the results obtained for $r = 2.0$ (tensile P) in comparison with the FE results of Haber and Hariandja [10], in which a separation zone occurs in the region directly below the concentrated load and slip occurs in the interval $0.85 \leq x/h \leq 1.6$ for the three BE approaches. For $r = -100$ (compressive P), contact is lost midway along the interface ($1.7 \leq x/h \leq 2.3$), as shown in 5, while a tiny portion of the interface directly below the concentrated load is sticking and the remainder on either side of the separation zone is slipping. It can be observed that the results from the BE approaches are in very good agreement with each other and with those of Haber and Hariandja [10].

**CONCLUSIONS**

The analysis of contact problems with frictional slipping using the boundary element software, BEACON, has been presented. The program offers three contact modelling approaches; the first based on node-on-node matching and the other two on independent discretisation of the contacting surfaces. The accuracy of these approaches has been demonstrated with the examples presented in which the results obtained are in very good agreements with each other and with other available solutions.
REFERENCES


