Frictional contact analysis using boundary element method

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Abstract

An efficient boundary element contact analysis based on an iterative and fully incremental loading technique is developed and implemented to solve non-linear frictional contact problems with conforming or non-conforming interfaces. This technique uses an updating process which allows the contact history to be followed from load-increment to load-increment. A traction extrapolation load scaling technique is used for each load increment to ensure that the unique combination of load, contact area and contact conditions is achieved for each potential contact node pair. In this paper, the problem of a tolerance-fit loaded pin in a cracked or uncracked lug is studied. The study shows that load transfer depends on the coefficient of friction. Stress intensity factors which are evaluated for a pair of cracks at the edge of the hole, from a $J$-integral at each load-step, are shown to depend on the coefficient of friction.

1 Introduction

Many different formulations for contact problems have been reported in both FEM and BEM over the last twenty years including Penalty methods [1]–[3], Lagrange Multipliers [4]–[5], Mathematical programming [6], Flexibility Matrices [7]–[8], Mixed or Hybrid methods [9]–[10] and Direct Constraint Methods [11]–[15] etc. A collection of references can be found in [16]. However, whilst the approaches are different, the basic ideas in solving the contact problems are the same; that is bodies are required to be tracked as they move relative to each other and when they finally come into contact, both the stress state and the size of the contacting region must be known. The contact state inside the contact interface as well as the normal and tangential traction distributions must be determined and the final deformed shapes must be found. In view of the complexity, the need for accurate numerical methods is apparent.

In this paper, the phenomenon of contact with friction is studied using the fully incremental loading technique [15, 17] for a tolerance-fit pin in a cracked
or uncracked lug. Stress intensity factors are evaluated for a pair of cracks at the edge of a hole, the boundary of which is loaded through a clearance-fit pin. Results which are obtained from J-integral, at each load-step, show that the presence of friction has a significant influence on the stress intensity factors, and hence on the residual strength and fatigue endurance of the structure.

2 Boundary Element Method (BEM)

For an elastic solid, the boundary integral equation is,

\[ C_{ij}(\mathbf{x}')u_j(\mathbf{x}') + \int_{\Gamma} T_{ij}(\mathbf{x}', \mathbf{x})u_j(\mathbf{x})d\Gamma(\mathbf{x}) = \int_{\Gamma} U_{ij}(\mathbf{x}', \mathbf{x})t_j(\mathbf{x})d\Gamma(\mathbf{x}); \]  

(1)

where \( \mathbf{x}' \) and \( \mathbf{x} \) are points on the boundary \( \Gamma \), \( C_{ij} \) is a constant depending upon the geometry at the point \( \mathbf{x}' \), \( u_j \) and \( t_j \) denote components of the boundary displacement vector and the traction vector respectively, and \( U_{ij} \) and \( T_{ij} \) are Kelvin's fundamental solutions. The boundary of the body is discretized and divided into a number of elements; over each element the variations of the geometry, displacements and tractions are described, in terms of nodal values and shape functions, in the usual way.

A system of simultaneous linear equations can be obtained using a standard collocation method [23] written as

\[ [H]\{u\} = [G]\{t\}. \]  

(2)

where \([H]\) and \([G]\) are coefficient matrices containing integrals of \( T_{ij} \) and \( U_{ij} \). The vectors \( u \) and \( t \) contain the nodal values of all the displacements and tractions. After application of the boundary conditions and rearrangement of the columns in \( H \) and \( G \) all the unknowns are passed to a vector \( \{e\} \) on the left hand side, giving

\[ [D]\{e\} = \{f\}; \]  

(3)

where \( \{f\} \) is known. Solution of the above system of equations may then be obtained by any standard method.

The relationship of the external load and the extent of the corresponding contact area is always unique, for monotonic loading, for a given problem subject to a given set of boundary conditions. However, this relationship is nonlinear in the presence of friction. The solution of the problem may have to be approximated by way of many small load steps. How the load steps are applied is critical in view of the nonlinear nature of the problem. In this paper, a fully incremental loading technique [15, 17] is used so that the unique relationship between a small increment in load and contact area can be accurately determined.

3 Fully Incremental Loading Technique

Derivation of the fully incremental loading technique is shown here by considering a discrete incremental load step \( \Delta P^m \) applied to a system which is initially in
equilibrium under a load $P^{m-1}$. The system will respond to the applied load by undergoing a small perturbation in displacement and traction everywhere on the boundary, to give a new equilibrium state. The changes in displacements $u_j$ and tractions $t_j$ can be defined as,

$$ u_j^m = u_j^{m-1} + \Delta u_j^m \quad \text{and} \quad t_j^m = t_j^{m-1} + \Delta t_j^m $$

and the external discrete load is now,

$$ P_j^m = P_j^{m-1} + \Delta P_j^m. $$

where $\Delta t_j^m$ and $\Delta u_j^m$ are the incremental changes in tractions and displacements respectively, due to the increment of load $\Delta P_j^m$. The incremental load required to increase the contact area by one linear element was calculated using a traction extrapolation technique [15, 17].

Substituting equation (4) into equation (1), and from the principle of superposition an integral equation in the $\Delta$ (incremental) variables can be obtained;

$$ C_{ij} \Delta u_j^m + \int_{\Gamma} T_{ij} \Delta u_j^m d\Gamma = \int_{\Gamma} U_{ij} \Delta t_j^m d\Gamma. $$

Numerical implementation of the BEM technique for two bodies ($A$ and $B$) in contact requires discretization of equation (6), one for each body, to give

$$ C_{ij}^\alpha \Delta u_j^\alpha + \int_{\Gamma_{nc}^\alpha} T_{ij}^\alpha \Delta u_j^\alpha d\Gamma^\alpha + \int_{\Gamma_{c}^\alpha} T_{ij}^\alpha \Delta u_j^\alpha d\Gamma^\alpha $$

$$ = \int_{\Gamma_{nc}^\alpha} U_{ij}^\alpha \Delta t_j^\alpha d\Gamma^\alpha + \int_{\Gamma_{c}^\alpha} U_{ij}^\alpha \Delta t_j^\alpha d\Gamma^\alpha, \quad \alpha = A \text{ or } B, \quad (7) $$

where $\Gamma_c^\alpha$ and $\Gamma_{nc}^\alpha$ denote contact and non-contact regions respectively.

For a numerical solution to the problem the boundary integral equations (7) on the boundaries of body $A$ and body $B$ are discretized separately. This produces two sets of equations (one for $A$ and one for $B$) given by

$$ C_{ij}^\alpha \Delta u_j^\alpha + \sum_{n=1}^{N^A} H_{ij}^\alpha \Delta u_j^{\alpha n} = \sum_{n=1}^{N^B} G_{ij}^\alpha \Delta t_j^{\alpha n}, \quad \alpha = A \text{ or } B, \quad (8) $$

where $N^A$ and $N^B$ are the total number of nodes for bodies $A$ and $B$ respectively. Two sets of simultaneous linear equations are obtained; they can be expressed in matrix form as

$$ [H]^\alpha \{ \Delta u \}^\alpha = [G]^\alpha \{ \Delta t \}^\alpha, \quad \alpha = A \text{ or } B, \quad (9) $$

The vectors $\{ \Delta u \}^\alpha$ and $\{ \Delta t \}^\alpha$ represent boundary values of displacements and tractions. At the potential contact regions, the two systems of equations share the boundary variables of the problem; that is the equations are coupled and must be solved simultaneously for any given combination of external load and contact conditions. For efficiency and flexibility, these common variables are not eliminated in the assembly of the equation matrix; but kept as unknowns of the problem; so that coupling of the systems of equations is achieved through the use of the compatibility and continuity equations at the node-pairs in the form of contact constraints. The constraints for the different contact conditions between node $a$ on $A$ and node $b$ on $B$ are as follows:
1. Stick or Partial Slip mode

\[ \Delta(t^a_n) - \Delta(u^b_n) = - \left[ (t^a_n)^{m-1} - (t^b_n)^{m-1} \right] \]
\[ \Delta(t^a_n) - \Delta(t^b_n) = - \left[ (t^a_n)^{m-1} - (t^b_n)^{m-1} \right] \]
\[ \Delta(u^a_n) + \Delta(u^b_n) = 0 \]
\[ \Delta(t^a_n) + \Delta(u^b_n) = g_0 - \left[ (u^a_n)^{m-1} + (u^b_n)^{m-1} \right] \equiv g_0^m \] \( (10) \)

2. Slip mode

\[ \Delta(t^a_n) - \Delta(t^b_n) = - \left[ (t^a_n)^{m-1} - (t^b_n)^{m-1} \right] \]
\[ \Delta(t^a_n) - \Delta(t^b_n) = - \left[ (t^a_n)^{m-1} - (t^b_n)^{m-1} \right] \]
\[ \Delta(t^a_n) = - \left[ (t^a_n)^{m-1} \right] \]
\[ \Delta(t^a_n) = - \left[ (t^a_n)^{m-1} \right] \] \( (12) \)

where \( g_0 \) is the initial separation. The previous total tractions and total displacements \((t^{m-1}, u^{m-1})\) which are already known make up the right-hand-sides; they are updated from load-step to load-step so that the contact history (contact status) as well as their geometric history (deformed shape) are retained as an integral part of the modelling process. The unknown incremental values on the left-hand-sides are expressed in the average local coordinate system. This formulation is not only versatile and accurate but is also efficient if the High Speed Solver [16] is used to speed up the solution process.

3. Separation mode

\[ \Delta(t^a_n) - \Delta(t^b_n) = - \left[ (t^a_n)^{m-1} - (t^b_n)^{m-1} \right] \]
\[ \Delta(t^a_n) = - \left[ (t^a_n)^{m-1} \right] \]
\[ \Delta(t^a_n) = - \left[ (t^a_n)^{m-1} \right] \]

4 Numerical Examples

The value of the stress intensity factor \( K \) for a pin-loaded lug assembly is influenced by the particular distribution of contact stresses around the perimeter of the contact interface between the pin and the hole bore. In this paper, the contact interaction between a pin and a lug is investigated. A rounded-end aluminium lug and a steel pin are used so that the Young’s modulus ratio is \( E_{pin}/E_{lug} = 3 \) with \( E_{pin} = 2.1 \times 10^5 \text{Nmm}^{-2} \). Two tolerances of \( \Delta R/R = 0.005 \) and \( 0.0075 \) will be examined. The problem is discretized with one region representing the pin and two regions representing the lug. Two equal length cracks, one from either side of the hole (see figure 1), are assumed. The problem dimensions are given by \( R=20\text{mm}, W=60\text{mm} \) and \( H=3W \). Both a cracked and an uncracked lug will be considered here for comparison. The cracklength is given by \( a/R=1.4 \). Finally, two coefficients of friction, namely \( \mu=0.0 \) and \( 0.4 \), will be examined.

The problem is solved in successive load increments until a maximum contact angle of 65° is reached. For ease of comparisons, contact solutions for cracked and uncracked lugs are presented side-by-side. Contact traction distributions and the
load history for frictionless ($\mu = 0$) case is presented in Figure 2, for a tolerance of $\Delta R/R = 0.005$. The corresponding frictional ($\mu = 0.4$) solutions are shown in Figure 3. Similar results have also been obtained for a tolerance-fit pin of $\Delta R/R = 0.0075$, and are shown in figures 4 and 5.

Differences in the normal traction $t_n$ distributions for cracked and uncracked lugs are apparent only when the contact angle becomes large. A larger contact angle corresponds to a larger external load exerted on the pin, which causes the cracks to open and change the deformation from the uncracked case.

The presence of friction has a significant effect on the interaction between the pin and the lug once a ‘large’ contact area is established. Examination of the relative tangential displacements for all the contacting node-pairs from load-step to load-step shows that when the contact area is small (initial contact at one point), the aluminium lug undergoes a greater deformation than the steel pin and tends to warp around the pin. This means the nodes on the lug slide away from the centre of contact. But, as more node-pairs are brought into contact with an increasing load the frictional forces now act over a much wider contact area; thus they have a much greater influence on the relative tangential displacement. At some load level, the increasing deformation of the steel pin can cause nodes already in contact to move away from the centre of contact; thus reverse slip occurs leading to the change in the sign of the tangential traction as shown in figures 3 and 5. The relative tangential displacements, showing the reverse slip, are shown in figure 6 as a function of the original nodal positions. However, no similar reverse slip was observed if a large squared-end plate was used [22]. This is attributed to the fact that a large squared-end plate is very much stiffer than the rounded-end lug.

In the presence of friction, node-pairs come into contact at a lower load than in the frictionless case. This effect has a direct consequence on the crack behaviour, because any deformation inside the contact region affects the crack opening displacement at the crack root. Consequently, stress intensity factors obtained reflect a similar behaviour to the total-load. In figure 7 the load history and the stress intensity factors $K_I$ and $K_{II}$ are plotted for various crack lengths against contact angles for the tolerance of $\Delta R/R = 0.005$ and 0.0075. Figure 7 also contains plots of $K_I/K_0$ and $K_{II}/K_0$ where $K_0 = (P^{\mu}/2W)\sqrt{\pi a}$.
Figure 2: Load history and Traction distributions

\[ \frac{\Delta R}{R} = 0.005, \mu = 0 \].

Figure 3: Load history and Traction distributions

\[ \frac{\Delta R}{R} = 0.005, \mu = 0.4 \].
Figure 4: Load history and Traction distributions 
($\frac{\Delta R}{R} = 0.0075, \mu = 0$).

Figure 5: Load history and Traction distributions 
($\frac{\Delta R}{R} = 0.0075, \mu = 0.4$).
Figure 6: Relative tangential displacements vs. Nodal positions.

5 Conclusions

A fully load incremental boundary element technique has been used to obtain stress intensity factors for cracks in pin-loaded lugs. The problem is non-linear because of the friction between the pin and the lug. It is shown that the distribution of both normal and tangential tractions at the interface between the pin and lug depend on the coefficient of friction and also on the degree of tolerance in the pin-fit. The stress intensity factors (mode I and mode II) for the crack length studied are less dependent on the friction and tolerance parameters but are no longer strictly linear with load. The effects of friction and tolerance are expected to be larger for very short cracks. Although not investigated here, the results will also depend on the relative stiffness of the pin and lug.
Figure 7: Load history and stress intensity factors for tolerance-fit pin in a cracked lug $\frac{\Delta R}{R} = 0.005$ and 0.0075.
References


