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The variational and numerical approach to contact with dry friction and non associated plasticity of soils: The implicit standard materials

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ABSTRACT

The work is devoted to numerical simulation of structures involving both contact with friction and plasticity of soils. It is well known that such material behaviours exhibit non associated laws. A new modelization of the law is proposed to recover flow rule normality for non standard material, in particulard for soil.

THE IMPLICIT STANDARD MATERIALS

A "good generalization" of the standard materials was proposed in [2-4], with preserves the notion of extremal couple and the convexity assumptions. For this family of materials, called implicit standard materials, the existance of a function b(x,y), convex with respect to x, when y remains constant, and convex with respect to y, when x remains constant, is postulated. The function b is said a bipotential if the following inequality is satisfied:

$$\forall\ (x',y'), \qquad b(x',y') \geq x'.y' \tag{1}$$

A couple (x,y) is said extremal if the equality is achieved in (1):

$$b(x,y) = x.y \tag{2}$$

Therefore, x and y are related by subdifferential mappings:

$$y \in \partial_x b(x,y), \qquad x \in \partial_y b(x,y) \tag{3}$$

An implicit standard material is a material of which the physical behaviours correspond to the extremal couples of a given bipotential. In other words, the relations (3) define the (multivalued) constitutive law of this material.

UNILATERAL CONTACT WITH COULOMB'S DRY FRICTION

Let Ω_1 and Ω_2 two bodies, initially in contact at a point, $\dot{u}$ the relative velocity at this point of Ω_1 with respect to Ω_2 and t the reaction subjected to Ω_1 from Ω_2.
In an usual way, Coulomb's cone of isotropic friction is defined by:

$$K_\mu = \{(t_n, t_t) \text{ such that } \|t_t\| \leq \mu t_n\} \tag{4}$$

unilateral contact with dry friction by the following differential inclusion:

$$-(\dot{u}_n + \mu\|\dot{u}_t\|, \dot{u}_t) \in \partial\Psi_{K_\mu}(t) \tag{5}$$

where $\Psi_{K\mu}$ is the indicatory function of K_μ.
It can be shown this constitutive law may be considered as an implicit standard material law [4]. For this, the following function is proposed:

$$b_c(-\dot{u}, t) = \psi_{R_-}(-\dot{u}_n) + \psi_{K_\mu}(t) + \mu t_n\|\dot{u}_t\| \tag{6}$$

It can be easily proved this function is a bipotential by verifying (1) and the extremal couple for this bipotential satisfy the law (5) by applying the relation (3b).

A NON ASSOCIATED FLOW RULE FOR SOIL MATERIALS

For Drucker-Prager model, the plastically admissible stresses belong to the following set (Fig.1):

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$$K_\sigma = \{(s_m, s) \text{ such that } f(s_m, s) = \|s\| - r(c - s_m \tan\varphi) \le 0\} \tag{7}$$

Where s_m is the hydrostatic pressure, s the stress deviator.

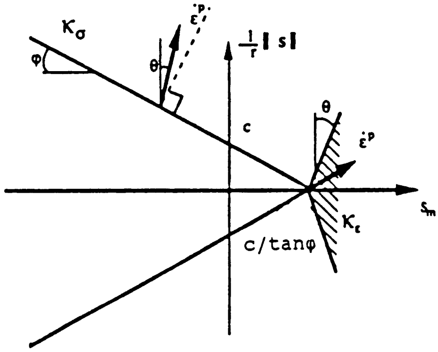

Fig.1. Rudnicki-Rice's non associated flow rule

A new formulation of Rudnicki-Rice's non-associated constitutive law [5] for soil material is introduced:

$$(\dot{e}^p_m + r(\tan\varphi - \tan\theta)\|\dot{e}^p\|, \dot{e}^p) \in \partial\Psi_{K_\sigma}(\sigma) \tag{8}$$

where $\dot{e}_m^p$ is the trace of the plastic strain rate tensor, $\dot{e}^p$ the plastic strain rate deviator. Then, the plastic strain rate belongs to the following set of admissible plastic strain rates:

$$K_\varepsilon = \{(\dot{e}^p_m, \dot{e}^p) \text{ such that } \dot{e}^p_m \ge r\tan\theta\|\dot{e}^p\|\}$$

This corresponds to an implicit standard material, because of introducing the following function:

$$b_p(\dot{\varepsilon}^p, \sigma) = \psi_{K_\varepsilon}(\dot{\varepsilon}^p) + \frac{c\dot{e}^p_m}{\tan\varphi} + \psi_{K_\sigma}(\sigma) + r(\tan\varphi - \tan\theta)\left(\frac{c}{\tan\varphi} - s_m\right)\|\dot{e}^p\| \tag{9}$$

It can be easily proved that this function is a bipotential by verifying (1) and the extremal couple for this bipotential satisfy the flow rule (8) by applying the relation (3b).

ELASTOPLASTIC EVOLUTION PROBLEM AND TIME INTEGRATION SCHEME

Considering the classical hypothesis of strain decomposition, the history of the couple (σ,ε) associated with the strain history $\varepsilon(t)$ is the solution $(\sigma(t),\varepsilon^p(t))$ of the following multivalued differential equation system of the first order:

$$S\,\dot{\sigma} + \dot{\varepsilon}^p = \dot{\varepsilon}(t), \qquad \dot{\varepsilon}^p \in \partial_\sigma\, b_p(\dot{\varepsilon}^p,\sigma) \tag{10}$$

$$S_c\dot{t} + \dot{u}^f = \dot{u}(t), \qquad \dot{u}^f \in -\partial_t\, b_c(-\dot{u}^f,t) \tag{11}$$

where S is the elastic compliance matrix, S_c is a fictitious contact compliance matrix introduced for numerical regularization of the law. In the following, the index 0 (resp. 1) is relative to the begining (resp. end) of the step and th symbol Δ to the increment. The implicit scheme gives:

$$\Delta\varepsilon = S\,\Delta\sigma + \Delta t\,\dot{\varepsilon}^p_1, \qquad \Delta u = S_c\Delta t + \Delta t\dot{u}^f \tag{12}$$

After some computations, the incremental law can be written as a normality law:

$$\Delta\sigma = \frac{\partial\Delta b_p(\Delta\varepsilon,\Delta\sigma)}{\partial\Delta\varepsilon}, \qquad \Delta t = \frac{\partial\Delta b_c(\Delta(-u),\Delta t)}{\partial\Delta(-u)} \tag{13}$$

VARIATIONAL AND FINITE ELEMENT FORMULATION

Let Ω be a structure of boundary S, in possible contact on the part S_2 of S subjected during a time increment to imposed body forces Δf, imposed surface traction increments $\Delta\bar{t}$ on the part S_1 of S, and imposed displacement increments $\Delta\bar{u}$ on the remaining part $S_0 = S - S_1 - S_2$ of the boundary. For this problem, usual notions of kinematically admissible (K.A.) strain increment field and statically admissible (S.A.) stress field can be defined. As proposed first in [2,3], the following new function, called bifunctional is introduced:

$$\Delta B(\Delta u,\Delta\sigma) = \int_\Omega [\Delta b_p(\Delta\varepsilon(u),\Delta\sigma) - \Delta f.\Delta u]\, d\Omega -$$

$$\int_{S_1} \Delta\bar{t}.\Delta u dS - \int_{S_0} \Delta t(\Delta\sigma).\Delta\bar{u} dS + \int_{S_2} \Delta b_c(-\Delta\dot{u},\Delta t)\, dS \tag{14}$$

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This definition allows to extend the classical calculus of variation to the implicit standard material. Because the bifunctional cannot be split anymore, the displacement and stress problems are coupled. So, it can be proved that a field couple ($\Delta u, \Delta\sigma$), exact solution of the boundary value problem and the constitutive law, is simultaneously solution of the following variational principles:

$$\underset{\Delta u^k K.A.}{Inf} \ \Delta B(\Delta u^k, \Delta\sigma) \quad and \quad \underset{\Delta\sigma^s S.A.}{Inf} \ \Delta B(\Delta u, \Delta\sigma^s) \tag{15}$$

For numerical applications, the method of displacement finite elements is used. The approximation of the displacement and strain increment field is defined by the relation:

$$\Delta u(x) = N(x)\ \Delta U, \qquad \Delta\varepsilon(x) = B(x)\ \Delta U$$
$$B(x) = grad_s\ N(x) \tag{16}$$

where ΔU is the unkown nodal displacement increment vector and $N(x)$ is a matrix of polynomial shape functions. Introducing the usual nodal force increment vector ΔF, the bifunctional (14) has the following discretized form:

$$\Delta B(\Delta U, \Delta\sigma) = \int_\Omega \Delta b_p(B\Delta U, \Delta\sigma)\, d\Omega - \Delta F^T \Delta U +$$
$$\int_{S_2} \Delta b_c(-N\Delta U, \Delta t)\, dS \tag{17}$$

Combining the structural equilibrium equations resulting from the minimum condition (20a) with the incremental law (13), it can be seen that the solution of the boundary value problem must verify the following equation systems:

$$\Delta\sigma_r = \frac{\partial \Delta b_p(B\Delta U, \Delta\sigma)}{\partial \Delta\varepsilon} - \Delta\sigma = 0\ ,$$
$$\Delta F_r = \int_\Omega B^T \Delta\sigma d\Omega - \Delta F = 0 \tag{18}$$

$$\Delta t_r = \frac{\partial \Delta b_c(-N\Delta U, \Delta t)}{\partial(-\Delta u)} - \Delta t = 0 \tag{19}$$

In principle, the equations (18a) and (19) should be satisfied anywhere. In pratical implimentation, the integrals are computed numerically by Gauss integration

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scheme. Then the local equations are only considered at Gauss points. Of course, the stress increments does not fulfill the local equilibrium equations but only the global ones (18b) in a mean sense with the weight functions B(x).

NUMERICAL PROCEDURE

To introduce the algorithm, it can be noted the similarity of this formulation with one of the key ideas of the large time increment or LATIN method proposed by Ladeveze. Using the terminology of this last approach, the couple (ΔU , $\Delta\sigma$), solution of the boundary value problem, is at the intersection of the non linear subspace Γ defined by the constitutive law (18a)(19) and the linear subspace A_d of the statically admissible solutions, defined by Eq. (18b) (Fig.2).

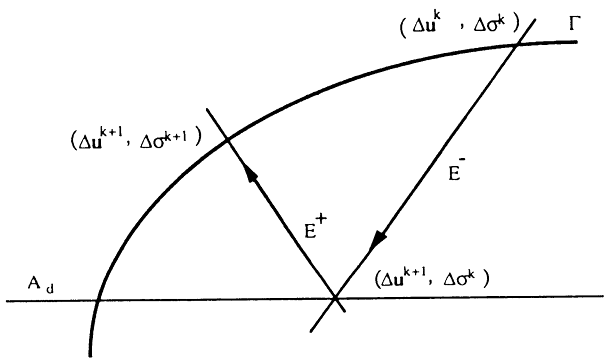

Fig.2. The LATIN method

The solving of (18a)(19) corresponds to the local stage, associated with an upwards search direction E^+, and the solving of (18b) to the global stage, with a downwards search direction E^-. For every step, the solving procrdure is based on an iterative procedure:

NUMERICAL APPLICATIONS

Rigid punching on semi-infinite space

A frictional elastoplastic material with Young's modulus E = 30000. psi, Poisson's coefficient ν=.3, a cohesion c = 10. psi and a friction angle φ = 20° is considered.

Talking into account problem symmetry and the limitation of the plastic yielding in the vicinity of the punch, only the portion of plane represented at fig.3 is discretized with the corresponding dimensions and boundary conditions indicated at this figure.

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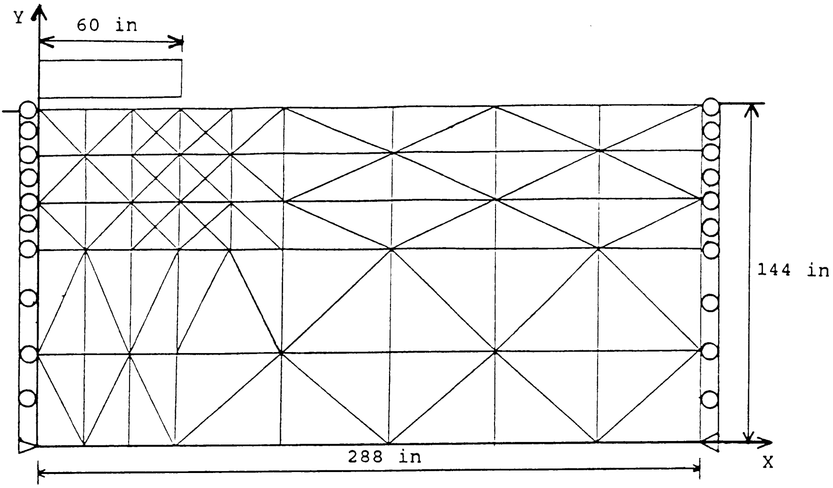

Fig.3. Geometry, loading ,boundary conditions and finite element model

The curves shown in Fig.4 give the comparizon between numerical calculus by finite element (FEM) with Mohr-Coulomb (M-C) criterion of plasticity, boundary element (BEM) with (M-C), Drucker-Prager (D-P). We can observed an excellent conformity with the different elastoplasticity curves.
In the same Figure, the load-displacement curves are compared for materials which differ only by the plastic dilatancy angle θ.

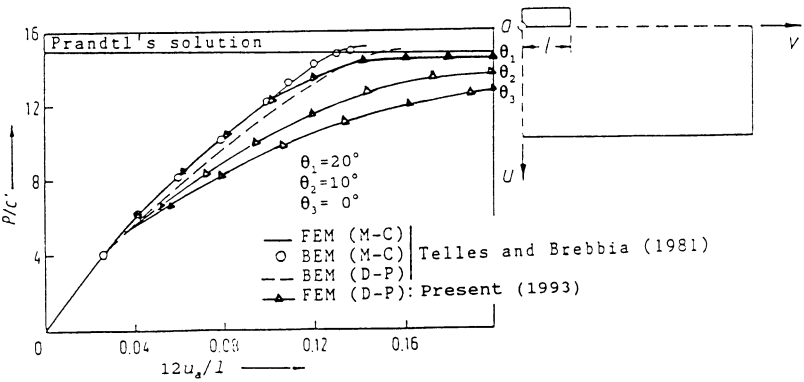

Fig.4. Load-displacement curves for strip footing problem

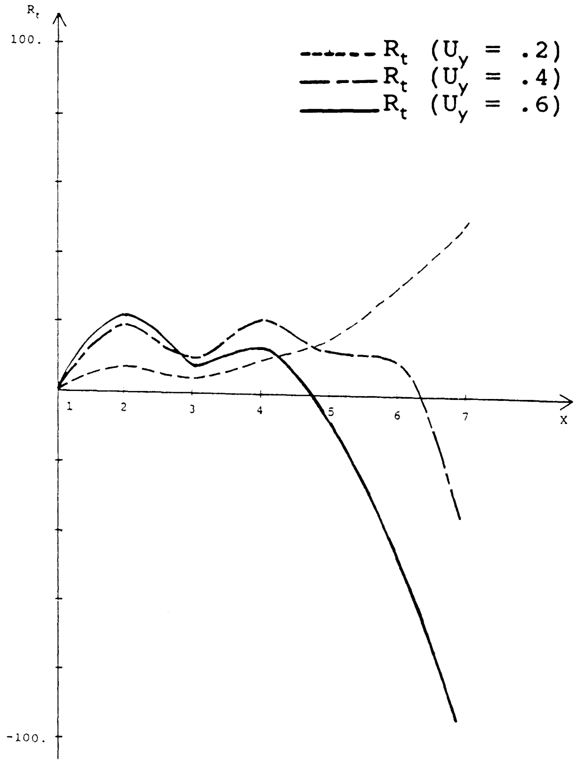

Fig.5. Tangential contact force
($\theta = 0°$, $\mu = 1.$)

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