Moving and rolling contact of 2D elastic bodies with defects using boundary element method

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Abstract

A scheme of boundary element method for moving contact of two-dimensional elastic bodies using conforming discretization presented by authors is briefly formulated at first. In the scheme both the displacement and the traction boundary conditions are satisfied on the contacted region in the sense of discretization. An algorithm to deal with the moving of the contact boundary on a larger possible contact region is presented. The algorithm can be generalized to rolling contact problem as well. Some numerical examples of moving and rolling contact of 2D elastic bodies with or without friction, including the bodies with a hole-type defect are given to show the effectiveness and the higher accuracy of the presented schemes. Finally, numerical examples of moving contact of 2D elastic bodies with a crack in the vicinity of contact region are presented as well.

1 A scheme of BEM for moving contact of 2D elastic bodies

1.1 Sub-domain approach of BEM with node-to-node scheme

Boundary element method is developed as a comparatively accurate and effective numerical method for engineering analysis in recent 40 years. For the linear problems, the BEM is usually more accurate and sometimes more effective than the FEM. For the nonlinear problems, if the corresponding differential equations are nonlinear, the advantage of BEM will be reduced. For the boundary-nonlinear problems, such as contact problem, BEM still preserve the advantage in comparison with FEM and other numerical methods\(^1,2\).

For the BEM of contact problem, sub-domain approach is usually applied\(^1\). A model for the contact of two 2-D elastic bodies is shown in Figure 1.
In the figure, $\Omega^i$, $i=1,2$ stand for the domains of two bodies, $\Gamma^{ni}$, $\Gamma^{ni}$ and $\Gamma^{ci}$ stand for the parts of the boundary of body $i$, where traction, displacement is given or contact is possible, respectively. The boundary integral equations in increment form can be written for each body:

$$
C_{ab}(p)\Delta u_{b}(p) = \int_{\Gamma^{ni}+\Gamma^{ni}+\Gamma^{ci}} U_{ab}(p,q)\Delta u_{b}(q)d\Gamma(q)
$$

$$
- \int_{\Gamma^{ni}+\Gamma^{ni}+\Gamma^{ci}} T_{ab}(p,q)\Delta u_{b}(q)d\Gamma(q)
$$

In matrix form they can be written as

$$
\begin{bmatrix}
A_{i1} & A_{i2} & A_{i3} \\
A_{i1} & A_{i2} & A_{i3} \\
A_{i1} & A_{i2} & A_{i3}
\end{bmatrix}
\begin{bmatrix}
\Delta U^i \\
\Delta T^i \\
\Delta T^{ic}
\end{bmatrix}
= 
\begin{bmatrix}
B_{i1} & B_{i2} & B_{i3} \\
B_{i1} & B_{i2} & B_{i3} \\
B_{i1} & B_{i2} & B_{i3}
\end{bmatrix}
\begin{bmatrix}
\Delta \bar{T}^i \\
\Delta \bar{U}^i \\
\Delta U^{ic}
\end{bmatrix}
$$

Where $\Delta U^i$, $\Delta T^i$ stand for the unknown boundary displacement and traction, $\Delta \bar{U}^i$, $\Delta \bar{T}^i$, the corresponding given values, and $\Delta U^{ic}$, $\Delta T^{ic}$, the displacement and traction on the possible contact boundary of body $i$.

After the variables on impossible contact boundary have been condensed, the equation only for the variables on possible contact region can be obtained:

$$
\tilde{A}_{i3}^{T} \Delta T^{ic} = \tilde{B}_{i3}^{T} \Delta U^{ic} + B_{i3}^{T}
$$

If the node-to-node scheme can be applied, the variables on the boundaries of possible contact region $\Gamma^{ci}$ should satisfy the following conditions:

$$
\Delta T^{ic} + \Delta T^{ic} = 0
$$

$$
\Delta U^{ic} - \Delta U^{ic} = D - \delta
$$

Where $\delta, D$ stand for initial and final gap of the node pairs for a load increment.

Finally, the equation to solve the variables $\Delta T^{ic}$ and $D$ on possible contact boundary can be obtained.
If the body I has been constrained, then
\[ E = \tilde{A}_{jj} + \tilde{B}_{jj} \left( \tilde{B}_{jj}^T \right)^{-1} \tilde{A}_{jj} \quad F = -\tilde{B}_{jj}^T \]

This equation can be formulated in a global Cartesian coordinate system or in a local coordinate system, normal and tangential along the boundary, either.

Because the contact problem is nonlinear, the above equation should be solved in iterative manner under the consideration of the variant contact state[3].

1.2 Node-to-movable-node scheme

To preserve much more characteristics of BEM, and to take advantage over FEM, a conforming discretization scheme in the possible contact region, node-to-movable-node scheme is proposed by authors[3]. The possible contact surface of each body is divided into special boundary elements. These elements consist of two fixed nodes at the ends and a movable node in the middle. Generally, a fixed node on one surface will not contact with a fixed node on the other surface. We take the point contact with the fixed node of another surface as the position of the movable node.

![Figure 2. The boundary elements and nodes on possible contact boundaries](image)

In Figure 2, it is illustrated the boundary elements and nodes at different step of contact procedure. Each element has two fixed end-nodes (black in the figure) and one movable middle node (white in the figure). If the element is in the state of separated, the movable node is located just at the middle point. While the element is contacted with a certain element, the movable node will be located at the contacted position with the fixed node of that element. On the separated boundary and contacted boundary, each element should be regarded as two sub-elements, from a black end-node to a white middle node. The boundary variables are linearly distributed on such sub-elements.

Under the assumption of small strain, for one element on the boundary of body I \((i')\), only one element \((j'')\) or two elements \((j''-\) and \(j''+)\) of body II may be contacted with it, provided the length of the boundary element of both bodies is taken as the same constant value. For the case that the element \(j''\) is contacted with the element \(i'\), the end-nodes are just contacted with the
end-nodes. Otherwise, the common end node of the elements \( j_{II^-} \) and \( j_{II^+} \) is contacted at an intermediate point of the element \( i_i^I \). At the same time, the end nodes of the element \( i_i^I \) are contacted at intermediate point of the element \( j_{II^-} \) and \( j_{II^+} \) respectively.

Because the linear interpolation functions is adopted on both sub-element, the contact state should be checked for each sub-element. The displacement is continuous at each node, but the traction can be discontinuous at each node, including end nodes and middle nodes. The interpolation functions are shown in Figure 3. Figure 3a shows the interpolation function for step \( N \) and \( N+1 \), and Figure 3b shows the interpolation function for the increment from time step \( N \) to \( N+1 \). The possible discontinuity of tractions is accomplished by the definition of \( t^- \) and \( t^+ \), two different variables at each boundary node. In this way, the moving contact can be simulated in point to movable point mode. The numerical methods for point-to-point mode can be generalized to this point to movable point mode without essential difficulty.

![Figure 3. The interpolation functions](image)

### 1.3 Moving of contact boundary on large possible contact region

In the case of moving or rolling contact, the possible contact region is usually much larger than the real contact boundary (RCB) at each time instant (Figure 4). In order to enhance the efficiency, the possible contact region can be divided into several sub-regions, the size of each one is taken as 3-5 times of the real contact boundary. At each instant, only one sub-region or two sub-regions are regarded as really contacted sub-region (RCSR), other sub-regions of possible contact region are non-contacted sub-region (NCSR), which can be simply regarded as traction free boundary, the boundary variables on it can be condensed. In this way, the equation (2) can be rewritten as

\[
\begin{bmatrix}
A_{11}^i & A_{12}^i & A_{13}^i & A_{14}^i \\
A_{21}^i & A_{22}^i & A_{23}^i & A_{24}^i \\
A_{31}^i & A_{32}^i & A_{33}^i & A_{34}^i \\
A_{41}^i & A_{42}^i & A_{43}^i & A_{44}^i
\end{bmatrix}
\begin{bmatrix}
\Delta U^i \\
\Delta T^i \\
\Delta T^{iC} \\
\Delta T^{iC}
\end{bmatrix}
= \begin{bmatrix}
B_{11}^i & B_{12}^i & B_{13}^i & B_{14}^i \\
B_{21}^i & B_{22}^i & B_{23}^i & B_{24}^i \\
B_{31}^i & B_{32}^i & B_{33}^i & B_{34}^i \\
B_{41}^i & B_{42}^i & B_{43}^i & B_{44}^i
\end{bmatrix}
\begin{bmatrix}
\Delta T^i \\
\Delta U^i \\
\Delta U^{iC} \\
\Delta U^{iC}
\end{bmatrix} 
\]

where indexes \( 3', 3'' \) indicate the block-matrices corresponding to the NCSR and RCSR, respectively. The size of the nonlinear equation to be solved incrementally is reduced to the size of \( A_{i_2}^i \). Only when the RCB moves across the sub-regions, a NCSR changes to RCSR and a RCSR changes to NCSR, the
degrees of freedom of the NCSR should be re-condensed. On the other hand, only the components corresponding to the movable nodes of the block-matrices related to RCSR should be updated step by step. In this way, the computational efficiency can be enhanced significantly.

The algorithm above mentioned is not difficult to generalize to the case of rolling contact. The main difference of these two cases is the description of the moving or rolling of one body along a boundary of another body, by means of certain given-displacement boundary conditions. The given-traction boundary conditions should be prescribed properly as well.

2 Numerical Examples

At first, the algorithm is checked using a test problem, a circular plate with constant vertical load is moving along a horizontal boundary of the foundation. The result obtained using the above mentioned algorithm is just the same as moving of the stress and deformation field of corresponding stationary contact problem. This means the algorithm is effective and accurate. But for such a case, it can be reduced into a much simpler problem, and it is not necessary to compute it as moving contact problem at all. In this paper some meaningful examples will be given: a circular plate with constant vertical load is moving along a horizontal boundary of the foundation with a hole-type defect near the boundary; a circular plate with a hole-type defect near the boundary is rolling along a horizontal boundary of the foundation. Furthermore, a circular plate with constant vertical load is moving along a horizontal boundary of the foundation with a crack near the contact boundary.

2.1 A semicircular plate moving along the foundation with a defect

The computational model is shown in Figure 5. A semicircular plate with radius $R = 10 \text{ mm}$ slides along the surface of elastic foundation, the size of the foundation in computation is taken as $L = 40 \text{ mm}$. There is a circular hole
located near the surface, its radius \( r = 0.125 \text{ mm} \), the distance between the hole-center and the surface is \( h \). The elastic constants of the plate and foundation are \( E = 4000 \text{ N/mm}^2 \), \( v = 0.3 \), the friction coefficient is \( \mu = 0.03 \). The vertical load applied on the semicircular plate is \( q = 0.25 \text{ N/mm} \).

![Diagram of a semicircular plate moving along the foundation with a defect](image)

**Figure 5.** A semicircular plate moving along the foundation with a defect

Some results for the case of \( h = 1 \text{ mm} \) are obtained. The tangential stress distribution along the hole-edge is shown in Figure 6 for different instant of moving contact, namely the difference of the horizontal coordinate of the centers of semicircular plate and circular hole \( b = 0.0, 0.5, 1.0 \) and \( 1.5 \text{ mm} \). The variation of the tangential stress at several nodes on hole-edge during the moving contact process is shown in Figure 7.

![Graph showing tangential stress distribution](image)

**Figure 6.** Tangential stress distribution along hole-edge for different contact location
Furthermore, the moving contact for the cases with different deepness of the hole-type defect has been investigated as well. The contact compression distribution along one-half of the contact boundary for different deepness of the hole-type defect is shown in Figure 8. For the case of $h = 1.00 \text{ mm}$, the corresponding result Hertz solution is also shown in the figure for comparison. All the above-mentioned five curves indicate the frictionless cases. For the case of $h = 0.15 \text{ mm}$ the result for the contact with friction (friction coefficient $\mu = 0.03$) is also shown in this figure. The figure shows that the contact compression distribution is affected by the deepness of the hole-type defect significantly. For the case with friction, the curve is not symmetrical as in the frictionless case.

Figure 8. Contact compression for different deepness of the hole-defect
2.2 A circular plate with a defect rolling along a foundation

The computational model is shown in Figure 9. A circular plate with radius $R = 10\text{ mm}$ rolls along the surface of elastic foundation, the depth of the foundation in computation is taken as $H = 40\text{ mm}$. There is a circular hole located near the surface, its radius $r = 0.125\text{ mm}$, the distance between the hole-center and the surface is $h$. On the plate there is a hole-type defect near surface as well. Its radius is also $r = 0.125\text{ mm}$, and the distance between the hole-center and the surface is $h$. The elastic constants of the plate and foundation are $E = 4000\text{ N/mm}^2$, $\nu = 0.3$. The vertical load applied on the plate is $q = 5\text{ N}$, and the plate is non-sliding rolling along the surface.

![Figure 9. Circular plate with a defect rolls along the foundation surface](image)

As an example, it is computed for the case $h = 1.0\text{ mm}$. The variation of the tangential stress at nodes on the hole-boundary of the plate is shown in

![Figure 10. Tangential stress variation at nodes on plate defect during rolling](image)
Figure 10, and the corresponding result for the defect of foundation is shown in Figure 11. These figures indicate that the significant alternating stress arises at the defects of both plate and foundation, and the alternating stress would result in fatigue of these kinds of components (for example, wheel and rail).

![Figure 11. Tangential stress variation at nodes on the defect of foundation during rolling](image)

2.3 A semicircular plate moving along the foundation with a crack

The computational model is shown in Figure 12. A semicircular plate with radius $R = 10\,\text{mm}$ slides along the surface of elastic foundation, the size of the foundation in computation is taken as $H = L = 100\,\text{mm}$. There is a horizontal crack located near the surface, its length $a = 7.2\,\text{mm}$, the distance between the crack and the surface is $h = 1\,\text{mm}$. The elastic constants of the plate and foundation are $E = 4000\,\text{N/mm}^2$, $\nu = 0.3$, and the loading $q = 0.25\,\text{N/mm}^2$.

![Figure 12. A semicircular plate moving along the foundation with a crack](image)
The variation of stress intensity factors of type I and II at the left and right tip (denoted by L and R) are shown in Figure 13.

Figure 13. Variation of stress intensity factor during moving contact

3 Concluding remarks

A scheme of BEM for moving and rolling contact of 2D elastic bodies using conforming discretization is presented. Some numerical examples of moving and rolling contact of 2D elastic bodies with or without friction, including the bodies with hole-type defect and crack are given. Because the contact boundary conditions can be satisfied in the sense of discretization, the results should be more accurate than non-conforming schemes, especially for the accuracy of local high stresses. In order to solve the complicated practical engineering problems, local material non-linearity should be taken into account, and the presented scheme should be generalized to the three-dimensional cases.

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References

