Accurate and computationally efficient two-dimensional unconditionally stable FDTD method

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Abstract

In this paper, an accurate and computationally efficient two-dimensional unconditionally stable finite-difference time-domain (2-D US-FDTD) method based on the Crank-Nicolson scheme is proposed. In particular, in the proposed 2-D US-FDTD method the field components are defined at only two time steps \( n \) and \( n+1 \); and the original time-dependent Maxwell’s equations of the Crank-Nicolson scheme are solved by introducing a proper intermediate value for a field component. Compared to the ADI-FDTD method, the US-FDTD method offers the following two advantages: i) the left-hand and right-hand sides of the original updating equations are balanced (in respect of time step) as much accurate as possible and, ii) only a single iteration that requires less number of updating equations is needed for the field development. The numerical performance of the proposed US-FDTD method over the ADI-FDTD algorithm is demonstrated through numerical examples.

1 Introduction

Recent publications [1,2] indicate that, by solving (some of) the updating equations implicitly, the Courant-Friedrich-Levy (CFL) stability condition required by the conventional finite-difference time-domain (FDTD) method can be totally removed. It has been demonstrated that the method is very useful for problems where very fine meshes with respect to the wavelength are needed for at least part of the computational domain [1,2]. Because of the alternating direction implicit
(ADI) technique adopted in [1,2], the field components used in the 2-D ADI-FDTD method have to be defined at three time steps \( n, n+1/2, \) and \( n+1. \) Therefore, two subiterations are needed for the field development (from time steps \( n \) to \( n+1 \)). Although defining the field components at three time steps makes the final form of the implicit updating equations simple, it consumes more computer memory. Also, the two subiterations slow down the speed of the FDTD simulation as more equations need to be updated. On the other hand, the left-hand and right-hand sides of the original updating equations of the ADI-FDTD method are not balanced in respect of time step [3]. Certainly, the unbalancing and the asymmetric effect limit the accuracy of the ADI-FDTD method.

To overcome the above drawbacks, in this paper an accurate and computationally efficient 2-D unconditionally stable FDTD (US-FDTD) algorithm based on the Crank-Nicolson (CN) scheme is proposed. It is shown that, by introducing a proper intermediate value for a field component, the original time-dependent Maxwell’s equations of the CN scheme can be reduced from a multi-dimensional implicit problem to a series of one-dimensional tridiagonal implicit problems. As a result, for the 2-D US-FDTD method only a single iteration (with less number of updating equations than those used in the ADI-FDTD method) is needed for the field development. In this paper, as an example, the 2-D TE wave is used to numerically demonstrate the advantages of the US-FDTD method.

2 Theory of the 2-D US-FDTD method

We use 2-D TE wave to demonstrate the theory of the 2-D US-FDTD method. If the electromagnetic field components are defined at time steps \( n \) and \( n+1 \) (note: the spatial locations of these field components are the same as those used in Yee’s scheme), then according to the Crank-Nicolson scheme the time-dependent Maxwell’s equations for the TE wave are given by:

\[
\frac{E_x|_{i+1/2, j+1/2}^{n+1} - E_x|_{i+1/2, j+1/2}^{n}}{\Delta t} = \frac{1}{2\epsilon\Delta y} \left( H_y|_{i+1/2, j+1/2}^{n+1} - H_y|_{i+1/2, j+1/2}^{n} + H_y|_{i+1/2, j+1/2}^{n} - H_y|_{i+1/2, j+1/2}^{n} \right)
\]

\[
\frac{E_y|_{i+1/2, j+1/2}^{n+1} - E_y|_{i+1/2, j+1/2}^{n}}{\Delta t} = \frac{1}{2\epsilon\Delta x} \left( H_x|_{i+1/2, j+1/2}^{n+1} - H_x|_{i+1/2, j+1/2}^{n} + H_x|_{i+1/2, j+1/2}^{n} - H_x|_{i+1/2, j+1/2}^{n} \right)
\]

\[
\frac{H_x|_{i+1/2, j+1/2}^{n+1} - H_x|_{i+1/2, j+1/2}^{n}}{\Delta t} = \frac{1}{2\mu\Delta y} \left( E_x|_{i+1/2, j+1/2}^{n+1} - E_x|_{i+1/2, j+1/2}^{n} + E_x|_{i+1/2, j+1/2}^{n} - E_x|_{i+1/2, j+1/2}^{n} \right)
\]

\[
-\frac{1}{2\mu\Delta x} \left( E_y|_{i+1/2, j+1/2}^{n+1} - E_y|_{i+1/2, j+1/2}^{n} + E_y|_{i+1/2, j+1/2}^{n} - E_y|_{i+1/2, j+1/2}^{n} \right)
\]

where \( \Delta t \) is the time step; \( \Delta x \) and \( \Delta y \) are the spatial increment in the \( x \) and \( y \) directions, respectively; and \( \epsilon \) and \( \mu \) are the permittivity and permeability of the media. Although the above equations can be directly solved with multidimensional implicit schemes, it is impractical as inverting and storing of large
matrices make the computation more expensive [4]. To avoid large matrix operations, the ADI [1,2] technique that reduces the multi-dimensional implicit problem to a series of one-dimensional tridiagonal implicit problems (with a two-step updating procedure) was developed. In this paper, an alternative way to solve the above equations (with a single updating procedure) is proposed.

To demonstrate the theory proposed in this paper, let us first define \( \hat{H}^{n+1}_x \) as the intermediate (and approximate) value of \( H^{n+1}_x \); and rewrite (1) and (3) as:

\[
\frac{E^{n+1}_{x|l+1/2,j}}{\Delta t} - E^n_{x|l+1/2,j} = \frac{1}{2\varepsilon\Delta y} \left( \hat{H}^{n+1}_{x|l+1/2,j+1/2} - \hat{H}^{n+1}_{x|l+1/2,j-1/2} + H^n_{x|l+1/2,j+1/2} - H^n_{x|l+1/2,j-1/2} \right) \tag{4}
\]

\[
\frac{\hat{H}^{n+1}_{x|l+1/2,j+1/2} - H^n_{x|l+1/2,j+1/2}}{\Delta t} = \frac{1}{2\mu\Delta y} \left( E^n_{x|l+1/2,j+1} - E^n_{x|l+1/2,j} + E^n_{x|l+1/2,j+1} - E^n_{x|l+1/2,j} \right) - \frac{1}{\mu\Delta x} \left( E^n_{x|l+1,j+1/2} - E^n_{x|l+1,j-1/2} \right) \tag{5}
\]

By substituting (5) into (4) and eliminating \( \hat{H}^{n+1}_x \), one has the final implicit updating equations for \( E_x \):

\[
\left[ 1 + \frac{(\Delta t)^2}{2\varepsilon\mu(\Delta y)^2} \right] E^{n+1}_{x|l+1/2,j} - \left[ \frac{(\Delta t)^2}{4\varepsilon\mu(\Delta y)^2} \right] \left( E^n_{x|l+1/2,j+1} + E^n_{x|l+1/2,j-1} \right) = \left[ 1 - \frac{(\Delta t)^2}{2\mu(\Delta y)^2} \right] E^n_{x|l+1/2,j} + \left[ \frac{(\Delta t)^2}{4\mu(\Delta y)^2} \right] \left( E^n_{x|l+1/2,j+1} + E^n_{x|l+1/2,j-1} \right) + \frac{\Delta t}{\varepsilon\Delta y} \left( H^n_{x|l+1/2,j+1/2} - H^n_{x|l+1/2,j-1/2} \right) - \left[ \frac{(\Delta t)^2}{2\varepsilon\mu\Delta x\Delta y} \right] \left( E^n_{x|l,j+1/2} - E^n_{x|l,j-1/2} - E^n_{x|l+1,j+1/2} + E^n_{x|l+1,j-1/2} \right) \tag{6}
\]

Once \( E_x \) is updated with (6), (2) and (3) can be used to derive the final updating equations for \( E_y \). In particular, the final implicit updating equation for \( E_y \) can be obtained by substituting (3) into (2):

\[
\left[ 1 + \frac{(\Delta t)^2}{2\varepsilon\mu(\Delta x)^2} \right] E^{n+1}_{y|l+1/2,j} - \left[ \frac{(\Delta t)^2}{4\varepsilon\mu(\Delta x)^2} \right] \left( E^n_{y|l+1/2,j+1} + E^n_{y|l+1/2,j-1} \right) = \left[ 1 - \frac{(\Delta t)^2}{2\mu(\Delta x)^2} \right] E^n_{y|l+1/2,j} + \left[ \frac{(\Delta t)^2}{4\mu(\Delta x)^2} \right] \left( E^n_{y|l+1/2,j+1} + E^n_{y|l+1/2,j-1} \right) - \frac{\Delta t}{\varepsilon\Delta x} \left( H^n_{y|l+1/2,j+1/2} - H^n_{x|l+1/2,j+1/2} \right)
\]
Eqs. (3), (6), and (7) are the actual final updating equations used in the 2-D US-FDTD method for the TE wave. Note that in the proposed US-FDTD method the intermediate value, \( \hat{H}_z^{n+1} \), is used only for deriving the implicit updating equation of \( E_x \), and it is no longer used after that. Actually, the \( H_z \) field component is updated with (3) after \( E_x \) and \( E_y \) have been updated. Thus, in the proposed US-FDTD method the field quantities are developed from time steps \( n \) to \( n+1 \) by solving three updating equations (but in the ADI-FDTD method six updating equations must be solved), in the order of (6), (7), and (3). It should be mentioned here that, although \( E_x^{n+1} \) appears at the right-hand side of the final \( E_y \) updating equation (see (7)), \( E_y \) can be updated once \( E_x \) is updated. In addition, (6) or (7) yields one-dimensional tridiagonal matrix systems and it can be easily solved with the approach presented in [5].

Finally, it is worth mentioning that, among the original updating equations (i.e., (2), (3), (4), and (5)) of the 2-D US-FDTD method, only (5) is not completely balanced in respect of time step. The reason for having such an unbalance in (5) is to derive the solvable implicit updating equation for \( E_y^{n+1} \) (i.e., (6)). This is also the reason why we said that the left-hand and right-hand sides of the original updating equations of the 2-D US-FDTD method are balanced, in respect of time, as much accurate as possible. Certainly, the field quantity \( \hat{H}_z^{n+1} \) (and others that depend on the value of \( \hat{H}_z^{n+1} \)) is less accurate. However, this will not affect the accuracy of the final simulated results much because \( \hat{H}_z^{n+1} \) is corrected when \( H_z^{n+1} \) is finally updated with (3).

### 3 Numerical validations

To demonstrate the accuracy and efficiency of the proposed 2-D US-FDTD method for the TE wave, the scattering property of a square dielectric object (with \( \varepsilon_r = 10.0 \) and located at the center of the computational space, as shown in Fig. 1) illuminated by a plane wave (excited with \( H_y^n \)) is studied. The time dependence of the source is a Gaussian pulse with a half-bandwidth of 1.5 GHz. A uniform mesh with cell spacing \( \Delta x = \Delta y = 2.5 \) mm is used; and the total mesh size is 101×101. The outer boundaries are closed with Mur’s 1st-order ABC [6].

Fig. 2 shows the scattering field (i.e., the difference between the total and incident fields) of the \( H_z \) component recorded at the observation point obtained with the US-FDTD and ADI-FDTD methods. It is worth emphasizing here that, to ensure the performances of the US-FDTD and ADI-FDTD methods are accurately compared, in this paper the scattering field is obtained from the
calculated total and incident fields with the advanced excitation approach [7]. In other words, in this paper the scattering field is not obtained from the total-field/scattered-field technique [8] since this technique still produces small numerical reflection or error. The steady state of the problem is reached when the total iteration time is up to 5.9 ns. The results shown in Fig. 2 are obtained with both the US-FDTD and ADI-FDTD methods for $\text{CFLN} = 4$ (where $\text{CFLN} = \Delta t / \Delta t_{\text{FDTD}}$, and in our case $\Delta t_{\text{FDTD}} = 5.9$ ps). For reference, the scattering field and scattering coefficient obtained with the conventional FDTD method (with $\text{CFLN} = 1$ and a sufficient large mesh size $801 \times 801$) is also plotted in Fig. 2. It can be seen from Fig. 2 that the accuracy of the numerical result obtained with the ADI-FDTD method is worse than that obtained with the US-FDTD method due to the unbalance of the ADI-FDTD method. On the other hand, to further confirm the stability of the US-FDTD method, the program was tested for 10,000 iterations with $\text{CFLN} = 40.0$, and no instability problem was observed.

![Diagram](https://example.com/diagram.png)

Figure 1: Geometric arrangement of the problem under study for TE wave.

To further demonstrate the accuracy and efficiency of the US-FDTD method, the scattering coefficient, calculated by $R = \text{DFT}(H_{i}^{\text{inc}})/\text{DFT}(H_{i}^{\text{minc}})$, caused by the square dielectric object is simulated. Fig. 3 shows the scattering coefficient, as a function of CFLN, obtained with the US-FDTD and ADI-FDTD methods. For reference, the scattering coefficient obtained with the conventional FDTD method (with $\text{CFLN} = 1$ and mesh size $801 \times 801$) is also plotted in Fig. 3. The results shown in Fig. 3 indicate that the scattering coefficient obtained with the ADI-FDTD method is less accurate than that obtained with the US-FDTD method, especially when the value of CFLN is bigger. Evidently, the result obtained with the ADI-FDTD method under the condition CFLN = 6 is no longer acceptable, which is understandable as the numerical error caused by the unbalance becomes stronger when the time step (or CFLN) gets bigger. The
above investigations certainly confirm the advantages of the proposed 2-D US-FDTD method.

![Graph](image)

**Figure 2:** Comparison of the scattering field obtained with US-FDTD and ADI-FDTD methods combined with Mur’s first-order ABC.

![Graph](image)

**Figure 3:** Comparison of the scattering coefficient obtained with US-FDTD and ADI-FDTD methods combined with Mur’s first-order ABC.

### 4 Conclusions

By defining the field components at only two time steps and balancing (in respect of time step) the left-hand and right-hand sides of the original updating equations
as much accurate as possible, a computationally efficient and accurate (compared to the 2-D ADI-FDTD method) 2-D US-FDTD method based on the CN scheme is developed. Especially, in the proposed US-FDTD method only a single iteration that requires less number of updating equations is needed for the field development. Therefore, the proposed 2-D US-FDTD method is, from both the implementation and computation points of view, also more efficient than the 2-D ADI-FDTD method. Furthermore, numerical examples demonstrate that the results obtained with 2-D US-FDTD method are more accurate that those obtained with the 2-D ADI-FDTD method.

References
