Acoustic scattering in fluid-solid problems: An inverse BEM formulation
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Abstract

In this paper a new boundary element method for sensitivity analysis of structures immersed in inviscid fluid is presented. The formulation developed utilizes the boundary integral equation for the Helmholtz equation for the external problem and the Cauchy-Navier equation for the internal elastic problem. An example is presented to demonstrate the accuracy of the proposed formulations.

1 Introduction

The NDT methods coupled with numerical techniques can provide efficient and accurate identification procedures. The two main numerical methods used in inverse analysis are the finite element and the boundary element methods (see for example Ref[1]). They can be coupled with optimization techniques to obtain the position and shape of the flaws by minimizing the error between numerical and experimental data at boundary sensor points.

In this paper a boundary element formulation based on the implicit differentiation method for sensitivity analysis of structures immersed in inviscid fluid and illuminated by harmonic incident plane waves is presented. The new formulation is then coupled to an optimisation technique to solve an inverse problem of flaw identification. The elastic object with an internal traction free flaw is assumed to be immersed in a inviscid fluid and illuminated by several known harmonic incident plane waves. An error function is introduced as a measure of the difference between the computed and observed acoustic pressures at sensor points of the boundary, and minimised to give the flaw position and shape. The numerical response is obtained enforcing normal equilibrium and compatibility between external Helmholtz equation and internal Cauchy-Navier equation.

2 Flaw Identification Method

The identification method proposed here is based on measurements (i.e. acoustic pressures or flux as well as displacements or tractions) taken at various sensor points, for different known incident plane waves. It was shown by [2] that, in these hypothesis, the scattering obstacle is uniquely determined.
2.1 Error function

If \( X_i \) is the \( i^{th} \) sensor node (\( i = 1, \ldots, NS \)) the measured result at this point for the \( d^{th} \) incident direction (\( d = 1, \ldots, D \)) is \( p_d^{(0)}(X_i) \) and the computed value for a set \( Z^k \) of design variables (iteration \( k \)) is \( p_d^{(k)}(X_i) \), the error at node \( i \), for the incident wave \( d \), is defined as

\[
\Delta_d^{(k)} p(X_i) = p_d^{(0)}(X_i) - p_d^{(k)}(X_i)
\] (1)

and the error function

\[
f(Z) = \frac{1}{D \times NS} \sum_{d=1}^{D} \sum_{i=1}^{NS} (\Delta_d^{(k)} p(X_i))^2
\] (2)

The optimization problem without any constraints on parameter values \( Z \) can be stated as follows

Find \( \{Z\} \) which minimises \( f(Z) \)

This minimisation is carried on updating the design variable vector \( Z \), at iteration \( k \), by

\[
Z^{(k+1)} = Z^{(k)} + \alpha^k s^{(k)}
\]

where the step parameter \( \alpha^k \) is chosen in order to minimise the error function in the search direction \( s^k \) and BFGS Quasi-Newton method is applied to obtain \( s^k \) ([4]).

The gradient of the error function is evaluated by

\[
\nabla f(Z)_m = \frac{\partial f(Z)}{\partial Z_m} = \frac{1}{D^2 \times NS^2 \times f(Z)} \sum_{d=1}^{D} \sum_{i=1}^{NS} \Delta_d^{(k)} p(X_i) \frac{\partial \Delta_d^{(k)} p(X_i)}{\partial Z_m}
\] (3)

where the terms

\[
\frac{\Delta_d^{(k)} p(X_i)}{\partial Z_m}
\]

are computed by implicit differentiation of the governing integral equations with the boundary conditions.

3 Boundary Element Method

In this section, the direct problem of the scattering of harmonic plane sound waves from an infinite cylinder in a fluid and the evaluation of design sensitivities of the flaw identification problem, are presented.

The repeated solution of the direct problem will be necessary to calculate the error function at every step of the optimization, while the design sensitivities will give the gradient of the error function at every step of the iteration process.

3.1 The Direct Problem

The model consists of coupling of two sets of integral equations: the steady state response of an elastic obstacle and the acoustic behaviour of the fluid,
in presence of an harmonic incident beam. The two equations are coupled by enforcing continuity of the normal components of velocities, and equilibrium of tractions on the external boundary $\Gamma_e$ plus traction free condition on the internal boundary $\Gamma_i$.

The external field pressure $p(x)$ is defined as

$$p(x) = p_{inc}(x) + p_{sc}(x)$$

where all the variables satisfy the Helmholtz equation and the scattering part satisfies also the Sommerfeld radiation condition at infinity [5].

The waves inside the scatterer are described by Cauchy-Navier equation with $u(x)$ and $t(x)$ denoting displacements and tractions respectively.

The compatibility and equilibrium conditions at the fluid-solid interface, and the traction free condition on the boundary of the cavity can be written as

$$q(x) = \rho_e \omega^2 u(x) \cdot n(x) \quad x \in \Gamma_e$$

$$t(x) = -p(x)n(x) \quad x \in \Gamma_e$$

$$t(x) = 0 \quad x \in \Gamma_i$$

(4)

where $n(x)$ denotes the inward normal on $\Gamma_e$, the flux $q(x)$ is the normal derivative of $p(x)$, $\rho_e$ the external density and $\omega$ the circular frequency.

3.1.1 Boundary Integral Equations

The boundary integral equation for the external problem governed by the Helmholtz equation [6] can be written as

$$c(\xi)p(\xi) + \int_{\Gamma_e} q^*(\xi, x)p(x)d\Gamma_e(x) = \int_{\Gamma_e} p^*(\xi, x)q(x)d\Gamma_e(x) + p_{inc}(\xi)$$

(5)

for the internal elastic problem as

$$c_{ij}(\xi)u_j(\xi) + \int_{\Gamma} T_{ij}(\xi, x)u_j(x)d\Gamma(x) = \int_{\Gamma} U_{ij}(\xi, x)t_j(x)d\Gamma(x)$$

(6)

where $\int$ stands for a Cauchy principal value integral, $p^*(\xi, x)$, $q^*(\xi, x)$ the fundamental solutions characterizing response of an harmonic point disturbance in an infinite acoustic domain; $U_{ij}(\xi, x)$, $T_{ij}(\xi, x)$ displacements and tractions respectively in the $i$ direction for an unit harmonic load acting along the $j$ direction in an infinite elastic plane [7].

3.1.2 Coupled System of Equations

Using isoparametric quadratic elements with shape functions $\phi^n(\zeta)$ and Jacobian $J^l(\zeta)$, the discretized boundary integral equations at node $\bar{\zeta}$ can be written as

$$c(\bar{\zeta})p(\bar{\zeta}) + \sum_{l=1}^{E \ell_{el}} \sum_{n=1}^{3} p^n \int_{-1}^{+1} q^*(\bar{\zeta}, x)\phi^n(\zeta)J^l(\zeta)d\zeta =$$

$$= \sum_{l=1}^{E \ell_{el}} \sum_{n=1}^{3} q^n \int_{-1}^{+1} p^*(\bar{\zeta}, x)\phi^n(\zeta)J^l(\zeta)d\zeta + p_{inc}(\bar{\zeta})$$

(7)
where $E_L$ denotes the number of elements on the external boundary and $E_L$ denotes the total number of elements. After applying the boundary conditions (4) the final linear system of equations (see [9]) can be written as

$$\begin{bmatrix}
H^e & 0 & 0 & -G^e & 0 \\
0 & H^i_{e,\Gamma_e} & H^i_{e,\Gamma_i} & 0 & -G^i_{e,\Gamma_e} \\
0 & H^i_{e,\Gamma_i} & H^i_{i,\Gamma_i} & 0 & -G^i_{i,\Gamma_i} \\
0 & \rho_e \omega^2 N^t & 0 & -I & 0 \\
N & 0 & 0 & 0 & I
\end{bmatrix} \begin{bmatrix}
p \\
u_{\Gamma_e} \\
u_{\Gamma_i} \\
q \\
t_{\Gamma_e}
\end{bmatrix} = \begin{bmatrix}
p_{inc} \\
o \\
o \\
o \\
o
\end{bmatrix}$$

where $H^e$ and $G^e$ contain the integrals involving $q^*$ and $p^*$ respectively and $H^i$ and $G^i$ contain the integrals involving $T_{ij}$ and $U_{ij}$ respectively; $N$ contains the components of the inward normal at every node of $\Gamma_e$ and $I$ is the identity matrix.

### 3.2 Design Sensitivity Analysis

The computation of the design sensitivities (used to compute the gradient of the error function) form a major part of the inverse method developed in the present work. In this context, design sensitivities are a measure of the sensitivity of external pressures and flux, internal displacements and tractions, to changes in design parameters.

This part presents an implicit differentiation design sensitivity analysis based on a singular formulation for quadratic isoparametric boundary elements. The procedure starts from the discretized form of the boundary integral equations (7), (8) and boundary conditions (4), differentiates them with respect to a design variable and solves the resulting system as in the standard system of the direct problem.

#### 3.2.1 Coupled System of Equations

At every step $k$ of the optimization process, the standard system is solved in order to compute the error function $f(Z^k)$.

To evaluate the gradient $\nabla f(Z^{(k)})$, it is necessary to differentiate the (4), (7) and (8). The differentiation of the boundary conditions on the external and internal boundaries gives

$$q_{m}(x) = \rho_e \omega^2 u_{m}(x) \cdot n(x) \quad x \in \Gamma_e$$
$$t_{m}(x) = -p_{m} n(x) \quad x \in \Gamma_e$$
$$t_{m}(x) = 0 \quad x \in \Gamma_i$$

(10)

It is worth noting that $c_{m} (\bar{\xi}) = 0$, $p_{inc,m}^*(\bar{\xi}, x) = 0$, $q_{m} (\bar{\xi}, x) = 0$, $J_{m}^i (\zeta) = 0$ and $p_{inc,m} (\bar{\xi}) = 0$ for every $\bar{\xi}, x \in \Gamma_e$ and denoting with $(\cdot), m$ the partial derivative with respect to $Z_m$, the differentiation of the discretized external integral
Computational Acoustics and Its Environmental Applications

Equations give

\[ c(\bar{\xi})p_{m}(\bar{\xi}) + \sum_{l=1}^{3} \sum_{n=1}^{3} p_{m}^{n} \int_{-1}^{+1} q^{*}(\bar{\xi}, x) \phi^{n}(\xi)J^{l}(\xi)d\xi = \]

\[ = \sum_{l=1}^{3} \sum_{n=1}^{3} q_{m}^{n} \int_{-1}^{+1} p^{*}(\bar{\xi}, x) \phi^{n}(\xi)J^{l}(\xi)d\xi \quad (11) \]

And the differentiation of the internal discretized integral equation gives

\[ c_{ij}(\bar{\xi})u_{j,m}(\bar{\xi}) + \sum_{l=1}^{3} \sum_{n=1}^{3} u_{j,m}^{n} \int_{-1}^{+1} T_{ij}(\bar{\xi}, x(\xi)) \phi^{n}(\xi)J^{l}(\xi)d\xi - \]

\[ = \sum_{l=1}^{3} \sum_{n=1}^{3} u_{j,m}^{n} \int_{-1}^{+1} U_{ij}(\bar{\xi}, x) \phi^{n}(\xi)J^{l}(\xi)d\xi = \]

\[ = \sum_{l=1}^{3} \sum_{n=1}^{3} u_{j,m}^{n} \int_{-1}^{+1} U_{ij,m}(\bar{\xi}, x) \phi^{n}(\xi)J^{l}(\xi)d\xi + \]

\[ + \sum_{l=1}^{3} \sum_{n=1}^{3} u_{j,m}^{n} \int_{-1}^{+1} U_{ij}(\bar{\xi}, x) \phi^{n}(\xi)J^{l, m}(\xi)d\xi - \]

\[ - \sum_{l=1}^{3} \sum_{n=1}^{3} u_{j,m}^{n} \int_{-1}^{+1} T_{ij,m}(\bar{\xi}, x(\xi)) \phi^{n}(\xi)J^{l, m}(\xi)d\xi - \]

\[ - \sum_{l=1}^{3} \sum_{n=1}^{3} u_{j,m}^{n} \int_{-1}^{+1} T_{ij}(\bar{\xi}, x(\xi)) \phi^{n}(\xi)J^{l, m}(\xi)d\xi \quad (12) \]

\[ c_{ij,m}(\bar{\xi}) = 0 \text{ on } \Gamma; \text{ if the design variables are chosen in order to keep smooth the geometry at any node of the design boundary.} \]

The above equations can be written in matrix form as

\[
\begin{bmatrix}
H_{e} & 0 & 0 & -G_{e} & 0 \\
0 & H_{i}\Gamma,\Gamma_{e} & H_{i}\Gamma_{e}\Gamma_{e} & 0 & -G_{i}\Gamma_{e}\Gamma_{e} \\
0 & H_{i}\Gamma_{e}\Gamma_{e} & H_{i}\Gamma_{e}\Gamma_{e} & 0 & -G_{i}\Gamma_{e}\Gamma_{e} \\
0 & \rho_{e}\omega^{2}N^{i} & 0 & -I & 0 \\
N & 0 & 0 & 0 & I
\end{bmatrix}
\begin{bmatrix}
p_{m} \\
u_{r,e,m} \\
u_{r,m} \\
q_{m} \\
t_{r,e,m}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 & -H_{i}\Gamma_{e,\Gamma_{e}}} & 0 \\
0 & -H_{i}\Gamma_{e,\Gamma_{e}} & -H_{i}\Gamma_{e,\Gamma_{e}} & 0 \\
0 & 0 & C_{\Gamma_{e},\Gamma_{e}} & 0
\end{bmatrix}
\begin{bmatrix}
u_{r,e} \\
u_{r} \\
t_{r,e}
\end{bmatrix}
\]

or rewritten in a more compact form as

\[ A X_{m} = A_{m} X \quad (13) \]
Notice the above equation has the same left hand side matrix, \( A \), as equation (9). The LU decomposition of \( A \) obtained for the solution of the direct problem can then be saved and reused for the solution of equation (13). This feature of the implicit differentiation formulation leads to considerable computational economy.

From (13) it is possible to compute all the sensitivities, i.e. the derivatives of the functions with respect the design variables, and then to update shape and position of the flaw using a optimization procedure. It is worth noting that at the new step \( k + 1 \) of the optimization, it is not necessary to re-evaluate the whole matrices \( A \) and \( A_{nm} \); in fact, it is enough to compute only the values involved in the \( H^i \) and \( G^i \) matrixes in which either the source node or the field node is on the internal boundary \( \Gamma_i \).

4 Shape Identification

An internal elliptical hole with parameters:

\[
\begin{align*}
x_1 &= -0.4 \\
x_2 &= -0.2 \\
\alpha &= 0.2 \\
b &= 0.1 \\
\phi &= -\frac{\pi}{6}
\end{align*}
\]

is identified using 16 sensor points, for different dimensionless wave numbers and different incident angles. The initial cavity is circular with radius 0.05 and centred at 0.5, 0.5. The material properties are given in Table 1.

Four incident waves (\( \alpha = 0, \alpha = \pi/2, \alpha = \pi, \alpha = 3/2\pi \)) are used to compare the results for \( kR = 0.5, 1, 2 \) while one incident wave (\( \alpha = 0, \alpha = \pi/2, \alpha = \pi, \alpha = 3/2\pi \) respectively) is shown for \( kR = 1 \) to measure the influence of \( \alpha \) on the identification procedure.

Figures 2 and 3 show the convergence of design variables and normalised error respectively in the case \( kR = 0.5, 1, 2 \) and four incident waves; it can be seen that after about 50 iterations the process has converged. The higher error for \( kR = 0.5 \) and \( kR = 2 \) show a lower capacity of these frequencies to identify the final cavity. For \( kR = 2 \) the optimization procedure gives a good solution \((-0.399, -0.200, 0.199, 0.0999, -0.520)\), but needs more constrains to avoid a higher number of local minima. For \( kR = 0.5 \) instead, the solution \((-0.432, -0.227, 0.169, 0.114, -0.010)\) presents a less accurate results.

For \( kR = 1 \) and four incident waves, the change in the hole shape and location after every 10 iterations are shown in figure 4

5 Conclusions

It has been shown that the boundary element method is an efficient technique for shape identification in two-dimensional bodies and scattering problems. The formulation of the method has been briefly described both from the theoretical and the numerical point of view. An example to demonstrate the accuracy of the method was also presented.
Brass cylinder
\[
\begin{array}{ll}
\text{Mass density} & \rho = 8500 \, \text{Kg/m}^3 \\
\text{Young's modulus} & E = 10.5 \cdot 10^{11} \, \text{Pa} \\
\text{Poisson's ratio} & \nu = 1/3 \\
\text{Radius} & R = 1.0 \, \text{m} \\
\end{array}
\]

Acoustic medium: water
\[
\begin{array}{ll}
\text{Mass density} & \rho = 998 \, \text{Kg/m}^3 \\
\text{Sound speed} & c = 1486 \, \text{m/sec} \\
\end{array}
\]

Table 1: Materials properties

Figure 1: Initial and final elliptical hole
Figure 2: Convergence of the design variables - \( kR = 1 \) and four incident waves.

Figure 3: Convergence of the normalised error for different dimensionless wave numbers - Four incident waves.
Figure 4: Identification of an elliptical hole
References


