A new formulation of the mixed finite element method for solving elliptic and parabolic PDE with triangular elements
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Abstract

For the Darcy flow model, we show how to produce a scheme with one unknown per cell starting from a mixed formulation discretized with the Raviart Thomas triangular element of lowest order. The aim here is to obtain a new formulation with one unknown per cell. In the first part, we describe the triangular mixed finite element (MFE) method used for solving Darcy’s and continuity equations. In the second part, we study the elliptic–parabolic problem. We describe the new formulation of the problem in order to use MFE with less unknowns obtained without any specific numerical integration. Along this work, we show that the new formulation can be seen as a general formulation which can be equivalent to the finite volume method in some particular cases.

1 Introduction

The idea of the mixed finite element is to approximate simultaneously the piezometric head P (L) and the velocity $q$ (L.T$^{-1}$). This approximation gives a velocity throughout the field and the normal component of the velocity is continuous across the inter-element boundaries. The drawback of an indefinite stiffness matrix is circumvented by hybridization. The system is solved in this case for one unknown, the pressure, per element.
edge. It has been noticed that for rectangular meshes, mixed finite elements of lowest order reduce to the standard cell-centered finite volume method (Chavent and Roberts, 1991) providing numerical integration is used. Recently, Cordes and Kinzelbach (1996) showed equivalence between mixed finite element and finite volumes, when general triangulation is used and without any particular numerical integration. This last connection was already noticed by Baranger et al. in 1994. However Ackerer et al. (1996) stated that such equivalence is restricted to divergence free velocity.

Mixed finite element (MFE) is more accurate but it uses more unknowns (number of edges) than the other methods. Hence the objective of this work is to reduce the number of unknowns for the MFE method using a new formulation in order to lead to a final system with the number of cells as unknowns. In this paper, one interests to a general triangular grid. In the first part, we describe briefly the triangular MFE method used for solving Darcy’s and continuity equations. In the second part, we present the elliptic-parabolic problem and the basis of the new formulation. Finally we present completely the new formulation and the connection between this formulation and finite volume (FV) method.

2 Triangular mixed finite element

In the lowest-order MFE formulation for triangular elements, the velocity vector is approximated with vector basis functions that are piecewise linear along both coordinate directions. Velocity $\mathbf{q}_E$, in any point inside element $E$ can be obtained by (Chavent and Roberts, 1991):

$$
\mathbf{q}_E = w_1 Q_1 + w_2 Q_2 + w_3 Q_3
$$

(1)

where $Q_i$ are the fluxes across the element edges $E_i$, and $w_i$ are the three vectorial basis functions of the Raviart Thomas element. The mass balance equation is discretized as following:

$$
\int_E \frac{\partial P}{\partial t} \, dv + \int_E \text{div} \, \mathbf{q}_E \, dv = Q_s
$$

(2)

This equation presents a finite volume aspect but finite volume schemes are usually expressed with the only unknowns $P$. With a finite difference schema in time we obtain:

$$
\frac{P^n - P^{n-1}}{\Delta t} + \frac{(Q_1 + Q_2 + Q_3)}{|E|} = \frac{Q_s}{|E|}
$$

(3)

where $s$ is the specific storativity of the porous material (\text{-}) and $Q_s$ the source or sink term (L^3T^-1).
Using the properties of the vectorial basis functions $w_i$, Darcy’s law written in a variationnal form leads to:

$$\sum_{j=1}^{3} Q_{E_{j}} \int_{E} (k^{-1} w_{j}) w_{i} = P_{E} - TP_{E_{i}}$$  \hspace{1cm} (4)

$P_{E}$ is the average potential head over $E$ and $TP_{E_{i}}$ are the average potential heads on each element edge $E_{i}$. Darcy’s law can then be written in the following matrix form:

$$\begin{bmatrix}
B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}
\end{bmatrix} \begin{bmatrix}
Q_{1} \\
Q_{2} \\
Q_{3}
\end{bmatrix} = \begin{bmatrix}
P - TP_{1} \\
P - TP_{2} \\
P - TP_{3}
\end{bmatrix}$$  \hspace{1cm} (5)

where $B_{ij} = \frac{1}{k} \int_{E} w_{i} w_{j}$

As shown by Cordes and Kinzelbach (1996) if we define $r_{ij}$ as the edge vector from node $i$ toward node $j$, $L_{ij}$ as its length ($L_{ij} = \|r_{ij}\|$), and by applying the scalar product, $r_{ij}r_{ik} = \frac{1}{2} \left( L_{ij}^{2} + L_{ik}^{2} - L_{jk}^{2} \right)$, we find:

$$B = \frac{1}{48k|E|} \begin{bmatrix}
3L_{12}^{2} + 3L_{13}^{2} - L_{23}^{2} & -3L_{12}^{2} + L_{13}^{2} + L_{23}^{2} & L_{12}^{2} - 3L_{13}^{2} + L_{23}^{2} \\
-3L_{12}^{2} + L_{13}^{2} + L_{23}^{2} & 3L_{12}^{2} - L_{13}^{2} + 3L_{23}^{2} & L_{12}^{2} + L_{13}^{2} - 3L_{23}^{2} \\
L_{12}^{2} - 3L_{13}^{2} + L_{23}^{2} & L_{12}^{2} + L_{13}^{2} - 3L_{23}^{2} & -L_{12}^{2} + 3L_{13}^{2} + 3L_{23}^{2}
\end{bmatrix}$$  \hspace{1cm} (6)

One notices that:

$B_{11} + B_{21} + B_{31} = B_{12} + B_{22} + B_{32} = B_{13} + B_{23} + B_{33} = L$

System of equations (5) gives then:

$L(Q_{1} + Q_{2} + Q_{3}) = 3P - (TP_{1} + TP_{2} + TP_{3})$  \hspace{1cm} (7)

Defining $S = |E|L - \frac{S}{\Delta t}$ and $F = LQ_{s}$ the substitution of equation (7) in the mass balance equation (3) gives:

$$P^{n} = \frac{1}{S + 3} \left( TP_{1} + TP_{2} + TP_{3} + F + S P^{n-1} \right)$$  \hspace{1cm} (8)

Using (8), equation (5) can also be solved for the corresponding fluxes across the element edges.

Defining $L_{1}^{-1} = -|E|L^{-1}/k$, $\zeta = \frac{L_{1}^{-1}}{3} - \frac{L_{1}^{-1}}{S + 3}$ and $b = \frac{L_{1}^{-1}}{S + 3} \left( F + S P^{n-1} \right)$, we find:

$$Q_{i} = a_{i1} TP_{1} + a_{i2} TP_{2} + a_{i3} TP_{3} - \frac{kb}{|E|} \text{ for } i = 1, 2, 3$$  \hspace{1cm} (9)
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where

\[ a_{11} = -\frac{k}{|E|} (r_{23} r_{23} - \zeta), a_{12} = a_{21} = -\frac{k}{|E|} (r_{23} r_{31} - \zeta), \]

\[ a_{13} = a_{31} = -\frac{k}{|E|} (r_{23} r_{12} - \zeta), a_{22} = -\frac{k}{|E|} (r_{31} r_{31} - \zeta), \]

\[ a_{23} = a_{32} = -\frac{k}{|E|} (r_{12} r_{31} - \zeta), a_{33} = -\frac{k}{|E|} (r_{12} r_{12} - \zeta) \]  \( (10) \)

3 The new formulation for the elliptic-parabolic problem

The goal is now to eliminate the state variables on edges with the help of a new variable associated to elements for a general case (steady state or transient flow with or without sink/source terms). This new variable \( H \) is defined such as the 3 fluxes have the following expression

\[ Q_i = \xi_i (H - \beta_i TP_i) + \gamma_i \quad \text{for } i = 1,2,3 \]  \( (11) \)

and such as \( H = \pi_1 TP_1 + \pi_2 TP_2 + \pi_3 TP_3 \).  \( (12) \)

\( H \) doesn't represent necessarily the average potential head on the element. Replacing (12) in (11), the coefficients \( \gamma_1, \gamma_2, \gamma_3, \pi_1, \pi_2, \pi_3, \xi_1, \xi_2, \xi_3, \beta_1, \beta_2 \) and \( \beta_3 \) are defined by equalling the MFE scheme (9). \( \gamma_1, \gamma_2, \gamma_3 \) are then given by

\[ \gamma_1 = \gamma_2 = \gamma_3 = -\frac{k}{|E|} b \]  \( (13) \)

The 9 coefficients \( (\pi_1, \pi_2, \pi_3, \xi_1, \xi_2, \xi_3, \beta_1, \beta_2 \) and \( \beta_3 \)) are obtained by solving the system (obtained by equalling (9) and (11)) constituted by equations which are not linearly independent. Therefore we assume one of the unknowns, \( \beta_1 \), is a given parameter noted in the following \( \tau \). The resolution of the system give then completely the 9 coefficients.

In the case of elliptic problem (steady state flow without sink/source terms), we have \( \zeta = 0 \) and if we take \( \tau \) equal to 1, our results are in agreement with those obtained by Cordes and Kinzelbach (1996). In this restrictive case \( H \) can be interpreted as the head at the circumcenter of the cell.
4 The MFE method with one unknown per element

In this part we construct the final system to solve with one unknown (H) by cell. We consider an element E with three adjacent elements (Figure 1). We use the continuity of pressure and fluxes between adjacent elements and we give the relation between the new variable H and the average potential head on the element.

The combination of equation (7) with equation (8) gives:
\[ L(1 + \frac{S}{3})(Q_1 + Q_2 + Q_3) + \frac{S}{3}(T_{P_1} + T_{P_2} + T_{P_3}) = F + SP^{n-1} \] (14)

Inserting (11) in the equation (14) leads to:
\[ (1 + \frac{S}{3}) (\sum_{i=1}^{3} \xi_i) H + \sum_{i=1}^{3} \left( \frac{S}{3L} - (1 + \frac{S}{3}) \xi_i \beta_i \right) T_{P_i} + (1 + \frac{S}{3}) \sum_{i=1}^{3} \gamma_i = \frac{F}{L} + \frac{S}{L} P^{n-1} \] (15)

If we note \( \lambda = (1 + \frac{S}{3}) \sum_{i=1}^{3} \xi_i \) and \( \lambda_i = \left\{ \frac{S}{3L} - (1 + \frac{S}{3}) \xi_i \beta_i \right\} \), equation (15) becomes:
\[ \lambda H + \sum_{i=1}^{3} \lambda_i T_{P_i} = \frac{F}{L} + \frac{S}{L} P^{n-1} + (3 + S) \frac{k_b}{|E|} = 0 \] (16)

This relation written for element E leads to:
\[ \lambda^E H_E + \lambda_1^E T_{P_1} + \lambda_2^E T_{P_2} + \lambda_3^E T_{P_3} = 0 \] (17)

For this element, if we define \( b_i^E = \xi_i^E \beta_i \) and \( \chi^E = -\frac{k_E}{|E|} b_E \), the fluxes across the element edges are:
\[ Q_i^E = \xi_i^E H_E - b_i^E T_{P_i} + \chi^E \] (18)

The continuity of fluxes and the continuity of pressure between two adjacent elements A and B (figure 1) lead to a relation which gives:
\[ T_{P_i}^A = T_{P_i}^B = \frac{\xi_i^A H_A + \xi_i^B H_B + \chi^A + \chi^B}{b_i^A + b_i^B} \] (19)

We can then expressed the flux between two elements A and B, function of \( H_A \) and \( H_B \) for the general case (steady state, transient flow with or without sink source term):
If we note: $b_{AB} = b_i^A + b_i^B$, substitution of equation (19) in (17) leads to the final system to solve:

$$
Q_{1}^{\Lambda} = \xi_{1}^{\Lambda} H_A - b_{1}^{\Lambda} \left( \frac{\xi_{1}^{\Lambda} H_A + \xi_{1}^{B} H_B + \chi^{\Lambda} + \chi^{B}}{b_{1}^{\Lambda} + b_{i}^{B}} \right) + \chi^{\Lambda} \tag{20}
$$

This system is solved with $H$ as unknowns. To resume, $H$ being calculated with system (21) we can compute average potential heads at element edges with equation (19). Equation (20) gives fluxes across element edges, and we finally obtain average potential heads inside each element by solving equation (3).

The new formulation (21) for a stationary problem, even with sink/source terms, leads to the resolution of a system of equation associated to a symmetric matrix. This matrix is positive definite if each angles of any element is lower than $\pi/2$.

In the case of steady state flow field with no sink/source terms, velocity vector is constant. Hence, the potential head varies linearly inside each element and $H$ may be interpreted as the potential head in the circumcenter of the element. We notice a complete analogy between the MFE and FV methods which gives us a new perception of the FV method. Notice that $H$ which is just a new variable is not the average potential head on the element but allows to calculate the average potential heads on edges, fluxes on edges and the average potential head on the element. Therefore, one can impose boundary conditions on the edges and $H$ may even be outside the element. A misinterpretation of the significance of $H$ can be at the origin of wrong statements. A rhombus being divided into two triangles by its longer diagonal, the circumcenters lie outside of their respective elements (figure 2). The flux across the common edge can be expressed by $Q_{AB} = L_{23} \frac{H_A - H_B}{L_A + L_B} \frac{1}{k_A + k_B}$ where $L_{23}$ is the length of the common edge 23, $L_A = L_{23} \frac{r_{12}r_{13}}{4|E_A|}$ and $L_B = L_{23} \frac{r_{42}r_{43}}{4|E_B|}$. 

$$
Q_{1}^{\Lambda} = \xi_{1}^{\Lambda} H_A - b_{1}^{\Lambda} \left( \frac{\xi_{1}^{\Lambda} H_A + \xi_{1}^{B} H_B + \chi^{\Lambda} + \chi^{B}}{b_{1}^{\Lambda} + b_{i}^{B}} \right) + \chi^{\Lambda} \tag{21}
$$

$$
H_A \left( \frac{\lambda_{1}^{A} \xi_{1}^{A}}{b_{AB}} + \frac{\lambda_{2}^{A} \xi_{2}^{A}}{b_{AC}} + \frac{\lambda_{3}^{A} \xi_{3}^{A}}{b_{AD}} \right) + \frac{\lambda_{1}^{B} \xi_{1}^{B}}{b_{AB}} H_B + \frac{\lambda_{2}^{B} \xi_{2}^{B}}{b_{AC}} H_C + \frac{\lambda_{3}^{B} \xi_{3}^{B}}{b_{AD}} H_D = -\lambda_{1}^{A} \left( \frac{\chi^{A} + \chi^{B}}{b_{AB}} \right) - \lambda_{2}^{B} \left( \frac{\chi^{A} + \chi^{C}}{b_{AC}} \right) - \lambda_{3}^{B} \left( \frac{\chi^{A} + \chi^{D}}{b_{AD}} \right)
$$
(\left|E_A\right| \text{ (respectively } \left|E_B\right|\text{)} being the area of the element } A \text{ (respectively } B\text{), } \mathbf{r}_{ij} \text{ being the vector } ij, \ k_A \text{ (respectively } k_B\text{) the hydraulic conductivity of the element } A \text{ (respectively } B\text{)). In the present case, } L_A \text{ and } L_B \text{ become negative. One could then explain that a head decrease from point } A \text{ to point } B \text{ corresponds in this case to a flux from } B \text{ toward } A. \text{ Furthermore, one could interpret this with local minima and maxima in the head field and nonphysical orientation of fluxes and velocities. In fact, there is no nonphysical orientation of fluxes because } H_A \text{ and } H_B \text{ are located outside their corresponding cells (at the circumcenters) and are not the average potential head inside their corresponding elements (figure 2). It appears that, in FV (and also of course for MFE), the triangulation has not to have properties formulated by Forsyth (1991) for standard triangular elements. The only thing is to calculate the average potential head over each element by using the values at the circumcenters.}

Conclusion

This paper show a general formulation of MFE(for transient, steady state flow, with or without sink/source terms) in order to solve the flow equation with less unknowns. The number of unknowns in this case is the number of cells instead of the number of edges. In case of steady state flow with no sinks or sources, this approach is equivalent to the FV method. This work shows that the FV method can be applied for whatever triangulation (circumcenter not necessarily inside the element).
References


