Abstract

Storm surge computations involve both an applied pressure field and a strong wind stress loading. The shallow water equation are commonly used for tidal simulations and have been thoroughly investigated for this type of loading. The storm surge computations on the other hand, have not been as extensively investigated. In this paper we examine the behavior of both the primitive and wave equation form of the shallow water equations and observe some deficiencies in mass conservation. Using a filtered primitive equation formulation and an extended derivative operator, improved solutions can be obtained.

1 Introduction

The solution of the shallow water equations for tidal simulations has been successfully accomplished by formulations based on both the primitive [1] and wave equation [2] forms. However, some recent experiments for storm surge computations, using both a prescribed surface elevation (due to the pressure drop at the center of a typhoon) and significant wind stress terms, resulted in solutions that appear to lose mass. Separating the two effects, it was determined that the difficulty arises due to the wind stress portion of the solution. To investigate this, a one dimensional analytic solution for an off shore wind was formulated and compared with the Generalized Wave Continuity Equation formulation (GWCE) and a Filtered Primitive
Equation (FPE) formulation. The primitive equations alone are unstable for this problem due to $2\Delta x$ waves. The FPE solution however is stable, but shows some mass balance loss. The GWCE solution is stable but shows a serious mass balance loss. One source of the loss in the GWCE formulation occurs due to the fact that the GWCE is based on a weak form of the continuity equation. The FPE equations on the other hand, are based on a strong form of the continuity equation. Another source of error occurs due to the land boundary condition. At boundary nodes, linear element only incorporate information over a single element whereas interior nodes incorporate information over two elements. As a result, numerical derivative operators are less accurate at the boundary giving rise to an accuracy loss at the boundary.

To improve on this, a special derivative operator was developed. This operator is based on the fact that gradients evaluated at the mid-side of elements, as an area weighted average of attached elements, demonstrate superconvergence. Using the derivatives at the mid-sides of elements, one can form a projection of the gradient to the nodes, that is of a higher order of accuracy. The projection is done entirely on linear triangles and does not require the use of quadratic element. Using the higher order derivative operator and the FPE formulation the results are significantly improved. The higher order derivative operator used in conjunction with the GWCE did not show any improvement.

2 Governing Equations

The full shallow water equations are based on a vertical integration of the Navier Stokes equations and mass conservation. The equations may be stated in Cartesian coordinates $(x, y)$ as:

\[
M_x = \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} + \frac{C_H}{H} U - f V + g \frac{\partial \zeta}{\partial x} - \frac{\tau_x}{H} = \rho
\]

\[
-\frac{1}{H} \left( \frac{\partial}{\partial x} \left( H A_H \frac{\partial U}{\partial x} \right) + \frac{\partial}{\partial y} \left( H A_H \frac{\partial U}{\partial y} \right) \right) - R(U - U) = 0
\]

\[
M_y = \frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + \frac{C_H}{H} V + f U + g \frac{\partial \zeta}{\partial y} - \frac{\tau_y}{H} = \rho
\]

\[
-\frac{1}{H} \left( \frac{\partial}{\partial x} \left( H A_H \frac{\partial V}{\partial x} \right) + \frac{\partial}{\partial y} \left( H A_H \frac{\partial V}{\partial y} \right) \right) - R(V - V) = 0
\]

\[
L = \frac{\partial \zeta}{\partial t} + \frac{\partial (HU)}{\partial x} + \frac{\partial (HV)}{\partial y} - RH = 0
\]

Equations 1 and 2 are the non conservative form of the $x$ and $y$ momentum equations and 3 is the continuity equation. $U$ and $V$ are the vertically averaged horizontal velocities and $\zeta$ is the change in surface elevation from
some initial datum. $H$ is the total depth $H = h + \zeta$, and $h$ is the bathymetric water depth. The other terms are: $g$ - gravitational constant, $f$ - Coriolis parameter $f = 2\Omega \sin \theta$, $\theta$ - latitude, $\Omega$ - angular velocity of the earth, $A_H$ - horizontal eddy viscosity, $\tau_x, \tau_y$ - $x$ and $y$ wind shear stresses, $\tau_x = C_w W^2 \cos \theta_w$, $\tau_y = C_w W^2 \sin \theta_w$, $C_w$ - wind shear coefficient, $W$ - wind velocity, $\theta_w$ - angle of wind measured positive from $x$ (east) axis, $C_b$ - bottom friction parameter $= g(U^2 + V^2)^{1/2}/C_z$, $C_z$ - Chezy coefficient, $R$ - internal fluid source, $U_o, V_o$ - velocity of the fluid source, $t$ - time, $n_x$ and $n_y$ - components of the outward unit normal. Appropriate boundary conditions are $\zeta = \zeta_p$ on a tidal boundary and $V_n = n_x U + n_y V$ or $T_x = n_x A_H \frac{\partial U}{\partial x} + n_y A_H \frac{\partial U}{\partial y}$, $T_y = n_x A_H \frac{\partial V}{\partial x} + n_y A_H \frac{\partial V}{\partial y}$ on a land or flow boundary.

The GWCE formulation is derived from the above equations by: 1) taking a time derivative of the continuity equation, 2) substituting the conservative form of the momentum equations $M_{xc} = UL + HM_x$, $M_{yc} = VL + HM_y$ and 3) adding the continuity equation multiplied by a penalty factor $G$. The formulation may be expressed as:

$$\frac{\partial L}{\partial t} + GL - \frac{\partial M_{xc}}{\partial x} - \frac{\partial M_{yc}}{\partial y} = 0 \quad (4)$$

This equation is typically solved for $\zeta$ while equations 1 and 2 are solved for $U$ and $V$. Equation 4 takes the form:

$$\frac{\partial^2 \zeta}{\partial t^2} + G \frac{\partial \zeta}{\partial t} - g \frac{\partial}{\partial x} \left( h \frac{\partial \zeta}{\partial x} \right) - g \frac{\partial}{\partial y} \left( h \frac{\partial \zeta}{\partial y} \right) = R \quad (5)$$

where all the nonlinear, velocity and load terms are contained in $R$.

### 3 Finite Element Derivative Evaluation

In a standard Galerkin formulation using linear triangles, the approximation of a derivative at a node involves the area weighted average of the derivatives of the elements surrounding a node such as node $o$ in figure 1a. The information that contributes to the derivative is restricted to the nodes shown by circles. We will refer to this as the standard derivative operator (SDO).

We now consider an extended derivative operator (EDO). Consider the derivative at the midpoint 2. This derivative can be computed from an area weighted average of the elements attached to point 2, i.e.,

$$\left\{ \frac{\partial \zeta}{\partial x} \right\}_{\text{midside}} = \left( A_o \left[ \frac{\partial N}{\partial x} \right] \{\zeta\}_o + A_p \left[ \frac{\partial N}{\partial x} \right] \{\zeta\}_p \right) / (A_o + A_p)$$

where $N$ are the linear triangle basis functions. The derivative at midpoint 2 now incorporates information from the node $p$, shown by the square.
Similarly, the derivative at midpoints 1 and 3 can be computed from the elements adjacent to these nodes. Given the derivatives at midpoints, we can write an expression for these values in terms of the nodal values of an element as shown in figure 1b. In terms of the natural coordinates $\xi$ and $\eta$, a linear interpolation for the derivative at any coordinate $\xi$ and $\eta$ may be written as:

$$\frac{\partial \zeta}{\partial x} = [N] \begin{bmatrix} \frac{\partial \zeta}{\partial x} \end{bmatrix}, \quad [N] = \begin{bmatrix} 1 - \xi - \eta & \xi & \eta \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial \zeta}{\partial x} \end{bmatrix}^T = \begin{bmatrix} \left( \frac{\partial \zeta}{\partial x} \right)_{node1} & \left( \frac{\partial \zeta}{\partial x} \right)_{node2} & \left( \frac{\partial \zeta}{\partial x} \right)_{node3} \end{bmatrix}$$

Substituting the $\xi$ and $\eta$ coordinates of the midpoints, we obtain an interpolation relation:

$$\begin{bmatrix} \frac{\partial \zeta}{\partial x} \end{bmatrix}_{midside} = [A] \begin{bmatrix} \frac{\partial \zeta}{\partial x} \end{bmatrix}_{nodes}, \quad [A] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Conversely we can obtain an extrapolation relation:

$$\begin{bmatrix} \frac{\partial \zeta}{\partial x} \end{bmatrix}_{nodes} = [A]^{-1} \begin{bmatrix} \frac{\partial \zeta}{\partial x} \end{bmatrix}_{midside}, \quad [A]^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

If we apply the above formula to all the elements attached to node $o$ and take an area weight average of the nodal derivative at each node, the information utilized in evaluating a nodal derivative, such as at point $o$, will incorporate information from the nodes marked by both the circles and squares of figure 1a. The SDO and EDO for the equilateral mesh (all sides of length 1) are compared for two boundary nodes in the table below.
Node 8: Last column is the % error for $\zeta = x^2 + y^2$

<table>
<thead>
<tr>
<th></th>
<th>SDO $\frac{\partial \zeta}{\partial x}$</th>
<th>EDO $\frac{\partial \zeta}{\partial x}$</th>
<th>SDO $\frac{\partial \zeta}{\partial y}$</th>
<th>EDO $\frac{\partial \zeta}{\partial y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$-\zeta_2 + \zeta_8$</td>
<td>$-25.0$</td>
<td>$\zeta_7 + 1.5\zeta_8 + .25\zeta_9 + .25\zeta_{19}$</td>
<td>$-12.5$</td>
</tr>
<tr>
<td>SDO</td>
<td></td>
<td></td>
<td>$\zeta_9 + .1443\zeta_{10}$</td>
<td></td>
</tr>
<tr>
<td>EDO</td>
<td>$-1.5\zeta_2 + .25\zeta_3 + .25\zeta_7 + 1.5\zeta_8 + .25\zeta_9 + .25\zeta_{19}$</td>
<td>$0$</td>
<td>$-19.25\zeta_{10}$</td>
<td>$0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+.7698\zeta_9 + .1925\zeta_{12}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+.0962\zeta_9 + 1.1547\zeta_{11} + .0962\zeta_{12} + .0962\zeta_{13}$</td>
<td></td>
</tr>
</tbody>
</table>

Node 11: Last column is the % error for $\zeta = x^2 + y^2$

<table>
<thead>
<tr>
<th></th>
<th>SDO $\frac{\partial \zeta}{\partial x}$</th>
<th>EDO $\frac{\partial \zeta}{\partial x}$</th>
<th>SDO $\frac{\partial \zeta}{\partial y}$</th>
<th>EDO $\frac{\partial \zeta}{\partial y}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$(\zeta_3 - \zeta_4 + \zeta_{10} - \zeta_{12}) / 3$</td>
<td>$0$</td>
<td>$.5774\zeta_9 - .5774\zeta_{19}$</td>
<td>$0$</td>
</tr>
<tr>
<td>SDO</td>
<td></td>
<td></td>
<td>$(3\zeta_3 - 3\zeta_4 - 1\zeta_9 + 3\zeta_{10} - 3\zeta_{12} - \zeta_{13}) / 3$</td>
<td></td>
</tr>
<tr>
<td>EDO</td>
<td>$\zeta_4 + .1925\zeta_{10} + .7698\zeta_9 + .1925\zeta_{12}$</td>
<td>$0$</td>
<td>$-22.22$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$+.0962\zeta_9 + 1.1547\zeta_{11} + .0962\zeta_{12} + .0962\zeta_{13}$</td>
<td></td>
</tr>
</tbody>
</table>

The errors in the $x$ and $y$ derivative for the doubly curved function $\zeta = x^2 + y^2$ are also shown in these tables. The EDO shows 1/2 the error of the SDO and is clearly better than a linear approximation but not as accurate as a quadratic approximation. Convergence is linear, that is, using elements with 1/2 the element side lengths (96 vs. 48 elements) the derivative errors are reduced by 1/2 for both operators.

4 Filtered Primitive Equation Formulation

The EDO has been incorporated into the filtered primitive equation formulation (FPE) and the generalized wave continuity equation formulation (GWCE) of the shallow water equations. Here we briefly discuss the FPE method since this method can be improved by the EDO formulation. Application of the Bubnov Galerkin method to equations 1→3 using nodal integration, yields an explicit system of equations

$$\frac{\partial U}{\partial t} = R_x, \ \frac{\partial V}{\partial t} = R_y, \ \frac{\partial \zeta}{\partial t} = R_z$$ (6)
These nonlinear equations are solved by a multi-step Taylor method. The terms \( R_x, R_y \), and \( R_z \) consist of nodal vectors \( V, \zeta, C_b, H, R, \tau_x, \tau_y \), and matrices \( M_n, DX, DY \). For linear triangles, \( M_n \) is a diagonal matrix with each entry one third of the sum of the element areas attached to a node. \( DX \) and \( DY \) are sparse matrices that produce the SDO or EDO derivatives at the nodes, i.e. \( \frac{\partial \zeta}{\partial x} = DX \ast \zeta \).

Since the Galerkin method for the primitive equations suffers from \( 2\Delta x \) oscillations, the surface elevations are periodically filtered, not necessarily at each time step. The filtered solution is obtained from equation 7.

\[
K_L \zeta' = P_L
\]
\[
K_F = \left\{ \begin{array}{c}
K_f \\
m_c \\
0
\end{array} \right\}
\]
\[
P_L = \left\{ \begin{array}{c}
P_F \\
V_c
\end{array} \right\}
\]
\[
\zeta' = \left\{ \begin{array}{c}
\zeta \\
\lambda
\end{array} \right\}
\]

The matrix \( K_F \) is a weighed \((W)\) second order derivative operator, \( \phi \) = linear basis functions, \( W=1 \) or nodal depth). The right hand side is constructed from a current \( \zeta \) vector and a Lagrange multiplier \( \lambda \) that is introduced to enforce mass conservation. The details of this formulation along with a dispersion analysis of the method are given in [1]. It has been shown that this filtering allows one to use a strong solution of the primitive equations rather than a weak solution as in the GWCE formulation. The stationary \( K_L \) matrix is reduced once and occasionally applied (e.g. every 8 time steps) producing very accurate solution for tidal simulations. This method avoids the selection and tuning of the penalty parameter "\(G\)" in the GWCE formulation. The dispersion analysis shows that the FPE method is not "folded" whereas the GWCE formulation is folded for "\(G\)" necessary to achieve mass conservation.

5 Numerical Results and Conclusions

We consider the problem when wind stress \((\tau_x, \tau_y)\) is the only loading.

![Figure 2. FE channel mesh. \(\tau_y\) wind load. Node 75 for \(\zeta(t)\) plots.](image-url)
Unlike tidal simulations, there are no prescribed surface elevations. All boundary conditions are on the velocity, typically requiring to zero velocity normal to the boundary. As a test problem, we consider a channel as shown in figure 2.

The linearized channel equations with constant coefficients and no friction ($\frac{\partial \xi}{\partial t} + h \frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial t} + g \frac{\partial \xi}{\partial x} = \frac{\tau_x}{h}$) have the solution[3]:

\[
\begin{align*}
\nu &= \sum_{p=0}^{\infty} a_p \cos \left( \left( \frac{1}{2} + p \right) \frac{\pi y}{l} \right) \sin \left( \left( \frac{1}{2} + p \right) \frac{\pi c}{l} t \right) \\
\zeta &= \frac{\tau_y}{gh} y - \frac{h}{c} \sum_{p=0}^{\infty} a_p \sin \left( \left( \frac{1}{2} + p \right) \frac{\pi y}{l} \right) \cos \left( \left( \frac{1}{2} + p \right) \frac{\pi c}{l} t \right) \\
c &= \sqrt{gh} \, a_p = (-1)^p \frac{\tau_y c l}{g (h \pi)^2 (1 + 2p)^2} 
\end{align*}
\]

In these equations, $\pm y$ is the distance from the channel midpoint and $l$ is one half of the total channel length. At $y = \pm l$ the series vanishes and we have $\frac{\partial \zeta}{\partial y} = \frac{\tau_y}{gh}$, consistent with the $\frac{\partial u}{\partial t} + g \frac{\partial \xi}{\partial x} = \frac{\tau_x}{h}$ since $\frac{\partial u}{\partial t} = 0$ at $y = l$.

Figure 3 compares the GWCE and FPE solutions using the higher order derivative operator with the exact solution for the surface elevation at a land boundary (node 75).

![Figure 3](image)

**Figure 3** Exact, FPE and GWCE solutions.

Figure 3 illustrate that the filtered primitive formulation (FPE) using the extended derivative operator (EDO) and shows a close fit to the analytic solution. Using the EDO operator in the wave equation formulation (GWCE),
however, did not show any improvement. The GWCE formulation shows a damped behavior, that may be related to the \( G \frac{\partial \kappa}{\partial t} \) term of the GWCE formulation (see equation 5) and the weak formulation of the continuity equation. The conclusion is that, the extended derivative formulation, EDO, used in conjunction with the flittered primitive equation formulation, FPE, is a viable scheme to obtain more accurate solutions of the surface elevation at land boundaries. The source of the mass loss, apparent in of the damped solution of the GWCE is currently a topic of continued study.

References

