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Vertical velocity in an SWE model: A penalty approach for improved mass conservation

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Abstract

The vertical velocity is an important, though often neglected component of the three-dimensional velocity field. Indeed, the computation of the vertical velocity within the confines of a three-dimensional, finite element surface water model is a difficult task. Although the vertical speeds are often several orders of magnitude less than the horizontal speed at the same point, the horizontal length scales are often several orders of magnitude greater than the vertical length scales. Therefore, small vertical velocities may result in substantial velocity transport.

The vertical velocity is computed according to the first order continuity equation. However, two boundary conditions are available and the system of equations is therefore overdetermined. Previous work has considered four approaches to the solution of this overdetermined system of equations. One of these, the Vertical Derivative of Continuity (VDC) method involves solution of the second-order equation obtained by differentiation, with respect to the vertical coordinate, of the first order continuity equation. The VDC method demonstrated poor point accuracy as well as poor mass conservation properties. This paper examines an extension of the VDC method, the K method which invokes a penalty to improve mass conservation. Preliminary results demonstrating point accuracy as well as mass conservation properties are presented.

1 Introduction

Computation of vertical velocity within the confines of a three-dimensional, finite element surface water model is a difficult but important task. Although the vertical speeds are often three to four orders of magnitude less than the horizontal speed at the same point, the horizontal length scales are often several orders of magnitude greater than the vertical length scales. Therefore, small vertical velocities may result in substantial velocity transport.

Finite element models based upon the shallow water equations generally solve these equations sequentially: the surface elevation is computed first, then the horizontal velocity is determined. The final step is determination of the vertical velocity according to the three-dimensional continuity equation:

$$L \equiv \frac{\partial w}{\partial z} + \nabla \cdot \mathbf{V} = 0 \tag{1}$$

where $w(x, y, z)$ is the vertical velocity, $\mathbf{V}(x, y, z)$ is the horizontal velocity vector, ∇ is the horizontal del operator, (x, y) are the horizontal coordinates and z is the vertical coordinate, positive upwards, with $z = 0$ at the surface. Because the horizontal velocity is computed before the vertical velocity, (1) is a first order, ordinary differential equation. Two boundary conditions are also available:

$$\left[w = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\right]_{z = -h} \tag{2}$$

$$[w = i\omega\eta]_{z = 0} \tag{3}$$

where h is the bathymetric depth, ω is the radian frequency, η is the free surface elevation and i is the imaginary unit ($\sqrt{-1}$). Either (2) or (3) can serve as the boundary condition, but imposition of both overdetermines the system. No unique solution exists for an overdetermined system of equations.

2 Previous Results

Previous work [*Muccino et al.*, 1997] described four approaches to the solution of this overdetermined vertical velocity problem:

1. Traditional (TRAD) method: The overdetermined continuity system is solved by simply not enforcing one of the boundary conditions. However, unsatisfactory accumulation of error over the water column is often encountered with this approach.

2. The Vertical Derivative of Continuity (VDC) method: The first order continuity equation (1) is differentiated with respect to the vertical coordinate, z. The resulting second order differential equation is solved and both boundary conditions are enforced. The point accuracy and mass conservation properties of the VDC method are poor.

3. Least Squares (LS) method: The continuity equation (1) is discretized, both boundary conditions are imposed, and the overdetermined system of linear algebraic equations is solved by minimizing the Eulerian norm according to the theory of least squares. Point accuracy and mass conservation properties of the LS method are very good, but CPU and memory requirements are substantial.

4. Adjoint (ADJ) method: This method is based upon the adjoint approach of Bennett and McIntosh. While LS minimizes the residuals of the continuity equation in its *discrete* form and the two boundary conditions, ADJ minimizes the residuals of the continuity equation in its *continuous* form and the two boundary conditions. Point accuracy and mass conservation properties of ADJ are comparable to those of LS, and the CPU and memory requirements are minimal.

3 Parallel to wave equation formulation

The current research on a penalty approach for vertical velocity calculations in an SWE model in many ways runs in parallel with the development of the wave equation approach for finite element models of surface water flow. Historically grid scale oscillations seemed to be a characteristic of the solution to the shallow water equations using the finite element method. [*Lynch and Gray,* 1979] recast the shallow water equations into an equivalent form, the wave continuity equation (WE), and in the process suppressed the spurious oscillations in a physically correct way. As discussed in the previous section, the use of the continuity equation for computation of vertical velocities in an SWE model yields an overdetermined system of equations because it requires two boundary conditions for a first order differential equation. [*Lynch and Werner,* 1987] and [*Naimie and Lynch,* 1993] introduced the second order form known as VDC, thus eliminating the issue of overdetermination.

It was shown for WE in [*Kolar et al.,* 1994] and for VDC in [*Muccino et al.,* 1997] that mass conservation is a problem with both the WE and VDC methods. *Kinnmark* [1986] reformulated the wave continuity equation such that the mass matrix is time independent. The resulting equation is known as the generalized wave continuity equation (GWCE). The GWCE formulation has better mass conservation properties than the WE formulation [*Kolar et al.,* 1994]. The velocity calculation work developed in this paper, the K method, reintroduces the first order conservation of mass equation into the second order VDC form, using a free parameter K, much like GWCE reintroduces the conservation of mass equation using a free parameter G in the WE form. Mass conservation is improved for both the GWCE and K methods. There is a difference however between the GWCE and K methods, in that G has a temporal dimension and K has a spatial dimension.

4 Derivation of the K method

As suggested by *Lynch and Werner* [1987] and *Naimie and Lynch,* [1993] the VDC method involves solution of a second order equation resulting from differentiation of (1) with respect to z:

$$\frac{\partial L}{\partial z} \equiv \frac{\partial^2 w}{\partial z^2} + \frac{\partial}{\partial z}(\nabla \cdot \mathbf{V}) = 0 \tag{4}$$

Because the VDC differential equation is second order both boundary conditions (2) and (3) may be enforced. However, the restriction of the original continuity equation is compromised in favor of the boundary condition. The resulting mass conservation properties are poor [*Muccino et al.,* 1997].

To enforce mass conservation the K method is now introduced. The K method adds the continuity equation, multiplied by a free parameter K to (4):

$$\frac{\partial L}{\partial z} + KL \equiv \frac{\partial^2 w}{\partial z^2} + K\frac{\partial w}{\partial z} + K(\nabla \cdot \mathbf{V}) + \frac{\partial}{\partial z}(\nabla \cdot \mathbf{V}) = 0 \tag{5}$$

If K is set equal to 0 in (5) VDC results. The Galerkin finite element method is implemented to solve (5) using a one-dimensional discretization, from the bottom(z_1) to the surface (z_n). The weighted residual form of (5), after integration by parts is:

$$-\int_{\Omega} \frac{\partial w}{\partial z}\frac{\partial \varphi_i}{\partial z} dz + \frac{\partial w}{\partial z}\varphi_i\Big|_{\Gamma} + \int_{\Omega} K\frac{\partial w}{\partial z}\varphi_i dz = -\int_{\Omega} \frac{\partial}{\partial z}(\nabla \cdot \mathbf{V})\varphi_i dz - \int_{\Omega} K(\nabla \cdot \mathbf{V})\varphi_i dz \tag{6}$$

where φ_i are weighting functions, Ω represents the one-dimensional domain, and Γ is the boundary of Ω, specifically the surface and the bottom. Because the vertical velocity at both boundary nodes is specified, (6) needs to be written only for interior nodes (nodes 2 through $n-1$) such that the boundary term vanishes. A Galerkin formulation with linear Lagrange weighting functions yields:

$$w = \sum_{j=1}^{2} w_j \varphi_j \tag{7}$$

Substitution of (7) in (6) leads to:

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$$\sum_{\text{elements}}\left(\sum_{j=1}^{2}\left(w_j\int_{\Omega_e}\frac{\partial\varphi_j}{\partial z}\frac{\partial\varphi_i}{\partial z}dz+Kw_j\frac{\partial\varphi_j}{\partial z}\varphi_i dz\right)=\int_{\Omega_e}\frac{\partial}{\partial z}(\nabla\cdot\mathbf{V})\varphi_i dz+\int_{\Omega_e}K(\nabla\cdot\mathbf{V})\varphi_i dz\right)$$

where Ω_e is the elemental domain. Implementation of linear Lagrangian basis functions and assembly of the elemental matrices yields a system of linear equations with tridiagonal structure that can be solved efficiently.

5 Results

Preliminary results are presented for the quarter annular harbor test problem as shown in Figure 1. The boundaries at $r = r_1 = 4\text{x}10^4\text{m}$ and $\theta = 0, \pi/2$ are no-flow boundaries. The open boundary, located at $r = r_2 = 1\text{x}10^5\text{m}$ is forced by an M_2 tide with frequency 1.405×10^{-4}/sec and amplitude 0.1m. The horizontal grid contains 825 nodes and 1536 elements. The bottom of the harbor, as shown in Figure 1, is quadratic in r and constant in θ such that $h = r^2/H$ and $H = 1.6\times10^8\text{m}$.

The vertical velocities are calculated using a range of different K values, from $K = 0$ to $K = 1\times10^7$. The solutions are converged in the vertical. Two typical velocity profiles are shown in Figures 2 and 3. In all cases midrange values of K more closely approximates the analytical solution [*Muccino et al.,* 1997] than does VDC ($K = 0$). As an example, in Figure 4, a mass conservation residual (see *Muccino et. al.*, 1997) is calculated over several tidal cycles for the single element centered at $r = 70,800$ m and $z = -13.7$ m. The residual obtained for all K is a sinusoidal wave with a period of 12.42 hours, the same period as the forcing. The amplitude for $K = 0$ (VDC) and $K = 1\times10^7$ are however significantly larger than that of $K = 1$.

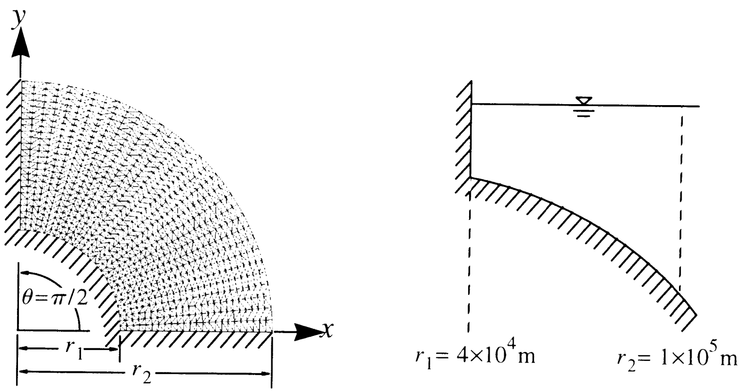

Figure 1. Section view of quarter annular harbor with opening at $r = r_2$ (left) and side view with quadratic bottom (right).

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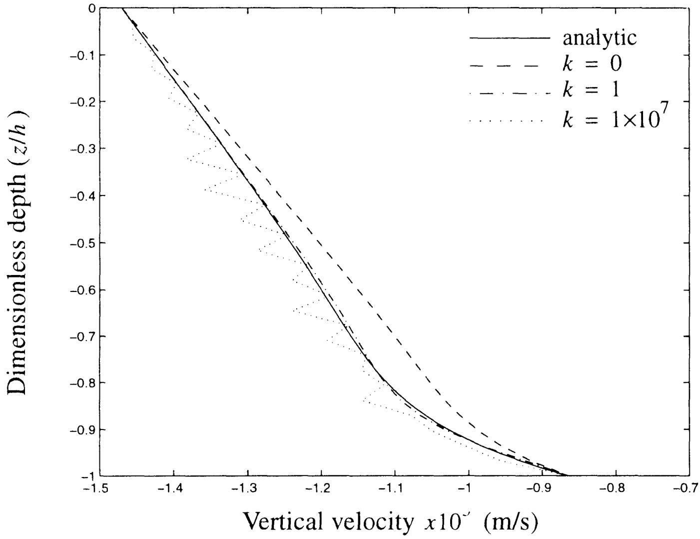

Figure 2. Comparison of vertical velocity profiles for analytic, $k = 0$, $k = 1$, and $k = 1\times10^7$ for $\lambda = 9.206 + 9.206i$ and $K = 2.836$ at $t = 11040\text{sec}$.

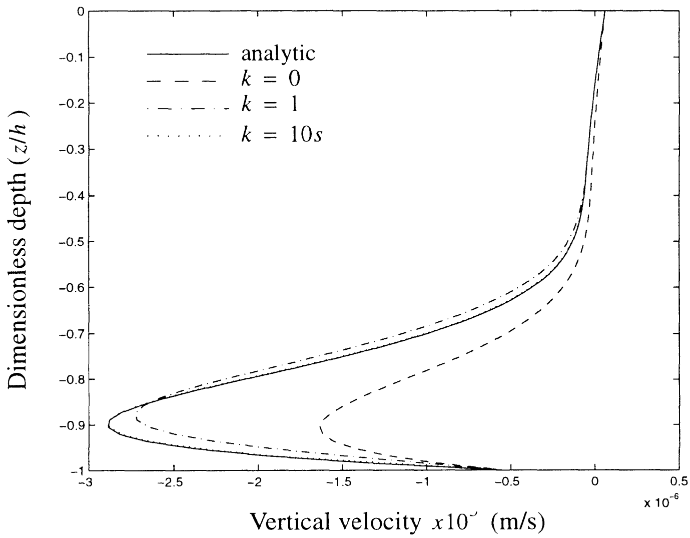

Figure 3. Comparison of vertical velocity profiles for analytic, $k = 0$, $k = 1$, and $k = 10$ for $\lambda = 6.627 + 6.627i$ and $K = 102.1$ at $t = 0$ sec.

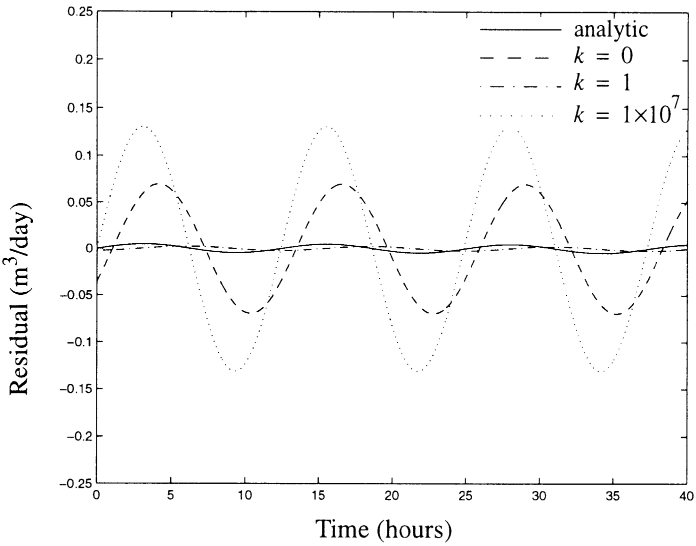

Figure 4. Residual vs. time for analytic solution and K method with $K = 0$, $K = 1$ and $K = 1\times10^7$ at $r = 70,800$ m and $z = -13.7$ m.

6 Conclusions

The K method has shown good point accuracy and good mass conservation properties in preliminary studies, for a large range of K values. There is a lower limit in K for which the method is useful, i.e. when it essentially starts to converge to the behavior of VDC, that is $K = 0$. There also appears to be an upper limit on K, above which the method gives less than optimal results. These findings are preliminary and further testing of the method is required. The method is computationally efficient in that it only requires integration of the divergence vector, assembly of a tridiagonal matrix, and the solution of the tridiagonal equation system. In addition the K method offers strong improvement in point accuracy and mass balance properties as well as convenient implementation in already existing implementations of the VDC method. The K method solves the overdetermination problem within the framework of numerical approximations of differential equations, which is conceptually simple and holds promise for many further applications.

7 Future Work

Some of the future work will include:

- more extensive application of the K method to test cases;
- application of the K method to a natural domain;
- determination of optimal K as a function of the parameters of the physical simulation region;
- further applications of the addition of the continuity equation to derivative models along the lines of G and K to enforce mass conservation - both from a physical and mathematical viewpoint;
- a priori estimates of the mass conservation properties of the mass imbalance control concept using analytical tools developed for the linear and non-linear analysis of differential equations;
- and extension of the method to nonlinear problems.

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