A Bayesian approach to electrical resistance tomography data inversion
Mengchun Yu and David E. Dougherty
University of Vermont, Burlington, Vermont 05405, USA
EMail: Mengchun.Yu@uvm.edu and David.Dougherty@uvm.edu

Abstract

Electrical resistance tomography can be an effective tool to delineate contaminated subsurface zones and to monitor subsurface remediation processes, provided accurate electrical resistivity images are achieved. Solutions from traditional regularization inverse methods are often unsatisfactory due to deleterious effects of the regularization term. In this work, datasets are obtained from multiple ERT configurations and assimilated in a Bayesian framework. The parameters are inverted by maximizing the a posteriori density. A modified total variation function is used to form the prior estimate. A new approach, successive partial variation relaxation, is developed to successfully minimize the deleterious effect of the subjective (prior) information.

1 Introduction

Electrical Resistance Tomography (ERT) is used to invert the measurements from electrodes in boreholes or/and on ground surface for the electrical resistivity of the soil. Pole-pole or dipole electrical sources are applied to electrodes and resulting electrical potentials or potential difference are measured on other electrodes. A set of data measurements developed by a particular spatial pattern of source locations is called an “ERT configuration”. ERT inversion is based on the principle that electrical energy dissipations along electrical current tubes depend on the resistivity distribution. The measured voltage is related to the average electrical energy dissipation between electrodes. In the ERT data acquisition, measurement noise is inevitable. However, ERT configurations can be designed to reduce
the ratio of the noise to signal [12]. The quality of ERT images depends on the strength of electrical sources, the accuracy of inverse algorithm, and the design of ERT configurations.

Because correlations exist between electrical and hydraulic properties of the soil [1], it is reasonable to utilize the ERT images in the subsurface environmental site characterization. Subsurface chemical contaminants may constitute electrically conductive anomalies that can be identified by ERT images. As a consequence, ERT could be an effective tool to leakage detection and remediation process monitoring. However, the low quality of previous ERT images limits the extensive application of the ERT technique. This work aims to advance the inverse technique so that ERT images can be generated with more accuracy.

ERT inversion is a nonlinear ill-posed estimation problem. The ill-posedness can be mitigated by imposing constraints (prior information), taking more measurements, and making an appropriate parameterization [3, 8]. In this work, all of these measures are brought together in a Bayesian framework. The Total Variation (TV) of parameters are considered as prior information. Data from multiple ERT configurations are utilized in the inversion.

2 Problem Specification

The partial differential equation governing the static (DC) electrical field in a conductive medium is given by

\[-\nabla_x \cdot \left( \frac{1}{\rho} \nabla_x \Psi \right) = \sum_j \delta(x - x_j) \quad x \in \Omega, \tag{1}\]

where \(\rho\) is the electrical resistivity, \(\Psi\) is the electric potential, \(i\) is the electrical source current on electrodes, \(\delta\) is the Direct delta function, \(x\) is the spatial position vector, and \(\Omega\) is the modeling domain. The 1\(^{\text{st}}\), 2\(^{\text{nd}}\), and 3\(^{\text{rd}}\) types of boundary conditions are imposed on boundaries.

ERT data inversion is considered to estimate electrical properties (parameters) \(p\) of porous media from measurement data \(z\). For this specific direct current ERT data inversion, \(p\) refers to the log electrical resistivity and \(z\) potentials at measurement points. The parameters \(p\) are represented by a time-invariant vector. Let an integer \(n\) be the
number of ERT configurations. Measurement data \( z = [z_1, z_2, \ldots, z_n] \) are acquired at ERT configurations 1, 2, \ldots, \( n \).

Consider the data \( z_k \) measured in the \( k \)th configuration for \( 1 \leq k \leq n \). The measurement data \( z_k \) are related through a nonlinear function to the parameter \( p \). Therefore,

\[
z_k = h(p) + v_k; \quad k = 1, 2, \ldots, n
\]

where the nonlinear function \( h \) is the ERT forward model, \( v_k \) is the model error vector arising in the data measurements, numerical discretization, and parameter zonation. Errors are assumed to be white and Gaussian with zero mean and covariance \( R_k \),

\[
E[v_k v_j^T] = R_k \delta_{kj} \quad \text{for } j, k = 1, 2, \ldots, n
\]

where \( \delta_{kj} \) is the Kronecker delta function. It is assumed that model error \( v_k \) are distributed independent of parameter \( p \),

\[
E[v_k p_j^T] = 0 \quad \text{for } j, k = 1, 2, \ldots, n
\]

### 3 Bayesian Approach for Data Inversion of Multiple ERT Configurations

Let the measurement data \( z \) and parameter \( p \) be random variables. It is evident that \( z \) and \( p \) are dependent on each other in the ERT problem. The conditional probability density function (pdf) of \( p \) given \( z \) is given by the Bayes’ rule as [4]

\[
f(p|z) = \frac{f(z|p)f(p)}{f(z)}.
\]

\( f(z) \) is the pdf of measurement data, which is the normalizing constant, and is given by

\[
c^{-1} = f(z) = \int f(z|p)f(p) \, dp.
\]

- **The likelihood function:** Assuming that the inverse of measurement error covariance matrix \( R_k^{-1} \) exists and using a Markov process, the likelihood density function \( f(z|p) \) in (5) can be written explicitly as

\[
f(z|p) = \prod_{k=1}^{n} \left[ (2\pi)^{-n_k/2} |R_k|^{-1/2} \right] \exp \left\{ -\frac{1}{2} \sum_{k=1}^{n} [v_k^T R_k^{-1} v_k] \right\},
\]
where \( R_k \) is the measurement error covariance matrix of the \( k \)th ERT configuration, \( n_k \) is the length of the vector \( v_k \). To reduce measurement uncertainties, data should be acquired repeatedly for each ERT configuration. As a result, the error vector \( v_k \) is an average of measurement errors for the \( k \)th ERT configuration.

- **A priori information:** In this work, the modified parameter total variation (PTV) function forms [11, 12] the \emph{a priori} density function \( f(\mathbf{p}) \). PTV assumes that nearby parameter values have a tendency to be similar. Based on this assumption, the \emph{a priori} density function can be defined as [11, 12]

\[
f(\mathbf{p}) = k_{PTV} \exp \left\{ -\frac{\beta}{2} F_t(\mathbf{p}) \right\}. \tag{8}
\]

We realize the disadvantage that the PTV prior density function (8) ignores parameter discontinuities in the model. If the optimal parameter is achieved through the maximization of the \emph{a posteriori} density, this disadvantage can be circumvented by employing a procedure of successive partial variation relaxation (SPVR) [12]. Thus a partially relaxed PTV function is given by

\[
F_t(\mathbf{p}) = 2 \sum_{i}^{N_p} \sum_{j}^{N_i} \left\{ \frac{|\nabla_x \mathbf{p} \cdot \mathbf{n}_{\Gamma_i^j}|}{\alpha + |\nabla_x \mathbf{p} \cdot \mathbf{n}_{\Gamma_i^j}|^2} \right\}. \tag{9}
\]

- **A posteriori density function:** Substituting (6), (7), and (9) into (5), the \emph{a posteriori} density function is obtained [12]

- **Optimal Bayesian estimation:** The \emph{a posteriori} density \( f(\mathbf{p}|z) \) provides the most complete description to the model parameter given available data. The Bayesian estimation then evolves to determine or approximate the \emph{a posteriori} density function. Depending on the criterion of optimality, one can compute an optimal parameter \( \hat{\mathbf{p}} \) from \( f(\mathbf{p}|z) \). There are two important criteria to obtain optimal estimation: Minimum Variance (MV) and Maximum \emph{A Posteriori} (MAP) estimates.

In the MV estimate the parameter \( \hat{\mathbf{p}}^{MV} \) is chosen, based on measurement data \( z \), so that error covariance \( E[(\mathbf{p} - \hat{\mathbf{p}})^T(\mathbf{p} - \hat{\mathbf{p}})|z] \) is minimized. Because ERT data inversion is a nonlinear problem, the computation of the conditional mean involves the evaluation of multidimensional integrals which presents a difficult task. Only simple problems with few parameters have been solved in the literature. It is
only feasible to seek numerical approximations of the \emph{a posteriori} density. Among methods to the approximation of the \emph{a posteriori} density function, the so-called extended Kalman filter (EKF) has been seen applications to practical problems \cite{5}.

The MAP estimate is used in this work. The MAP estimate of $\hat{p}^{MAP}$ is chosen based on observation data $z$, such that the \emph{a posteriori} density is maximized. The approach is to use optimization technique to locate the peak or mode of $f(p|z)$. The MAP estimate provides a convenient basis from which the maximum likelihood method, deterministic regularization method, and conventional least-squares can be derived. The MAP solution could be biased if the \emph{a posteriori} distribution is not symmetric about its mean. The MAP estimate is identical to the Kalman filter for linear problems with Gaussian \emph{a priori} \cite{10}. The maximum of the \emph{a posteriori} can be achieved by minimizing

$$
F(p) = \sum_{k=1}^{n} \{ v_k^T R_k^{-1} v_k \} + \beta F_i(p).
$$

Through (10), the problem of MAP is transformed into a problem of regularized least-squares. An error covariance matrix can then be derived by matrix differentiation \cite{6, 12}

\section{A Synthetic Example}

Figure 1: A vertical cross-section of the layered resistivity structure test model (from \cite{12}, with permission).
Layered resistivity structures are extremely difficult to be captured by the ERT image [2, 7]. To test the Bayesian approach developed in this work, a synthetic layered resistivity model is created. The model is a simplified version of a geoelectrical section at Applied Research Associates, Inc.'s Vermont site [9]. The modeling domain size is 10m×12m×13m (Fig. 1). 26 electrodes are installed in two boreholes and 6 electrodes on the ground surface.

Figure 2: ERT images resulting from synthetic data (from [12], with permission).

Noise-free datasets are created by recording the forward model responses on electrodes developed by various ERT source application patterns with a constant voltage of 360 V on the true resistivities. Noise-corrupted datasets are obtained by adding computer-
generated, normally distributed random noise with zero mean and variance 0.1. A volume weighted average resistivity 530 Ω·m is used as a starting value for all parameter zones. To obtain a satisfactory result, it is essential to use datasets of multiple ERT configurations. For noise-free data inversion, datasets of 10 ERT configurations are used with the TV weighting coefficient $\beta = 1.0$. Because noise causes the data inversion to be more unstable, 20 corrupted datasets are used and the more strict TV constraint is put to stabilize the inverse solution ($\beta = 5.0$).

The true image and inversion results are presented in Fig. 2. LaBrecque et al. [7] claimed that inverted images could not be equal to the true model even for noise-free data. However, this difficulty is overcome by the SPVR technique. The exact image is inverted for the noise-free data. The TV solution is similar to LaBrecque et al.'s solution. Due to the deleterious effect of the subjective prior information, the TV solution is not satisfactory.

For the corrupted datasets, the inverted images are less accurate. More SPVR iterations are required. The layered geologic structure is approximately identified. Examining the ERT configuration design, there are not enough electrical current paths through the midst of layers 3 and 4, which results in inaccurate images in these two layers.

5 Conclusion

A Bayesian approach is developed to invert electrical potential measurements for resistivity images. The SPVR technique effectively removes inaccurate constraints to inversions. Because this approach generates accurate resistivity images, it can be used to analyze effects of electrical source strengths and ERT configurations, which provides a guideline for ERT experiment design. This approach advances the ERT technique, which ultimately can be utilized as a cost-effective tool to subsurface characterization, monitoring of remediation processes, and detection of landfill and storage tank leakages.

Acknowledgments

This work was supported in part by NSF Grant EPS-9350540. The authors are grateful to Applied Research Associates, Inc. for providing CPT electrical measurement data.
References


