Phase diagram of fully developed drainage in porous media
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Abstract

A phase diagram of fully developed drainage in porous media is developed using concepts of Invasion Percolation in a Gradient. We show that the transition between stabilized displacement (where the conventional continuum description applies) and fingering is controlled by the change of the sign of the gradient of the percolation probability (from stabilizing to destabilizing). The transition boundary is described by scaling laws.

1 Introduction

The displacement (drainage) of a wetting (w) fluid in a porous medium by a non-wetting (nw) fluid has been analysed in detail in the past. For drainage in an $L \times L$ pore-network at velocity $q$, Lenormand\textsuperscript{1} identified three limiting patterns, Invasion Percolation\textsuperscript{2} (IP), Piston-like or Compact and Viscous Fingering\textsuperscript{3} (VF), and delineated their validity. As the displacement proceeds, however, spatial profiles will develop, and the overall displacement will be characterized by one of two possible global regimes, where there is a continuous transition between the limiting patterns. The first regime involves a frontal region with the structure of an IP cluster of extent $\sigma$, followed by an upstream region of increasing length with the characteristics of a
compact pattern. This Stabilized Displacement (SD) regime is where the conventional continuum description applies. The second regime is typically described as a VF-type pattern (for example of the DLA type). This, however, overlooks capillarity, which at sufficiently small scales should be comparable to viscous and should affect the properties of the fingers (as for example in stability studies). For the sake of generality, we will refer to this regime as Capillary-Viscous Fingering (CVF).

At present, the delineation of the validity of these regimes is not available. In two recent publications, we presented an approach based on Invasion Percolation in a Gradient, in which we recognized that the existence of the two different global patterns, depends on whether invasion is in a stabilizing gradient (IPSG) or a destabilizing gradient (IPDG), respectively. Here, we summarize the main results obtained.

2 Theory: Properties of developed regimes

Consider drainage at constant rate in a random porous medium represented as a network of pores \(L^{d-1} \times N\) with a pore throat size distribution with mean \(r_m\) and standard deviation \(\Sigma r_m\). If the displacement becomes a SD, there will be a frontal region, where the pattern is IP, of width \(\sigma\). The region is centered around the mean front position, \(X_c(t)\), which travels with velocity \(v\), and, in analogy with IPG, is defined as the place where the transverse average of the percolation probability \(p\) is equal to the percolation threshold, \(p(X_c) = p_c\). It is shown in [6] that use of IPSG results into the following equation for the determination of \(\sigma\)

\[
(\frac{b\sigma^{\zeta + \nu(1-D)}}{\nu} - M\sigma) \sim \frac{2\Sigma}{Ca_F} \sigma^{-\frac{1}{\nu}}
\]

where \(M\) is the viscosity ratio, \(M = \frac{\mu_w}{\mu_n}\), \(Ca_F = \frac{\nu \mu_n}{\gamma}\) is the capillary number at the front and \(b\) is a dimensionless constant. Exponents \(\nu\), \(\zeta\) and \(D\) correspond to the correlation length, the conductance and the mass fractal dimension of the percolation cluster, respectively. Upstream of the front there is a compact region, the transition to which can be described by a cross-over function.

Equation (1) admits a unique positive solution for all values of \(M\) and \(Ca_F\). At small \(Ca_F\), this solution approaches the power-law
asymptote $\sigma \sim \left(\frac{CaF}{2\Sigma}\right)^{-\frac{1+\xi+\nu(D-1)}{\epsilon}}$. The fact that a solution exists for all $M$ and $CaF$ also means that a fully developed SD exists (but is not necessarily stable) for all $M$ and $CaF$ (note that because of the existence of a solution, the LHS of (1) is positive, hence $p$ decreases in the direction of displacement). To establish whether this regime will actually develop, we must address its stability. Equivalently, this can be done by examining the development of the initial phase of the displacement, before a travelling-wave solution develops.

Before we proceed, we also briefly mention the CVF regime. In the context of IPG, this will develop when $p$ increases in the direction of displacement, in which case it will be IPDG. In analogy with [9] we expect that in such a displacement the fingers will be influenced by both viscous and capillary effects. It can be shown that in the case of large $M$, the problem becomes a standard IPDG problem with Bond number $B \sim -\frac{CaM}{2\Sigma}$, through which the average finger width can be directly obtained, namely $\sigma \sim \left(\frac{CaM}{2\Sigma}\right)^{-\frac{\nu}{\epsilon+1}}$. This result shows that the finger width will decrease with an increase in $Ca$ and $M$, eventually reducing to a thin finger of the size of a single pore (and where a DLA regime will emerge).

3 Delineation of the two regimes using IPG

Consider, now, the delineation of the validity of these regimes, in which case one needs to address the initial phase of the displacement. During this initial period, the displacement has an extent $\chi(t) \times L^{d-1}$, where $\chi$ is increasing with time. Its pattern will be of the IP type as long as $\chi(t) \leq \chi_{c}(Ca,M)$, where $\chi_{c}(Ca,M)$ is to be determined. When the pattern first departs from percolation (at $\chi = \chi_{c}$), the transition towards a fully developed displacement starts. The latter will either become a SD (with a compact region following an IP front) or a CVF regime, depending on whether at $\chi_{c}$, $p$ decreases or increases in the direction of displacement, respectively. To trace this transition we need, first, to identify $\chi_{c}$ and, second, to determine the sign of the gradient of $p$ at that point.

We define $\chi_{c}$ by following Lenormand and requesting that at $\chi_{c}$, we must have $\frac{\Delta N_{p}}{N_{p}} = \epsilon \ll 1$, where $N_{p}$ is the fraction of sites of the nw phase occupying an IP cluster. The scaling of the latter can be obtained from percolation theory, i.e. $N_{p} \sim (\text{const})(p-p_{c})^{\beta}$. It can then be shown that the following equation results for $\chi_{c}$.
the solution of which will be discussed below. This equation represents a generalization of two equations in [1] describing the IP-to-compact and IP-to-VF boundaries, respectively (which were determined in [1] in what amounts to a $\chi_\epsilon \times \chi_\epsilon$ lattice under the assumption that the pressure drop occurs only in one phase). Before we analyze (2), we must note that $c_1 \chi_\epsilon^{-\nu}$ actually represents the large-$\chi$ asymptote of the ratio $\frac{M \Delta P_{\text{tw}}}{\Delta P_{\text{tw}}}$ as shown in Fig. 1. At sufficiently small $\chi$, the curve approaches a constant value $M^* \sim 1$.

The solution of (2) depends on the sign of $c_1 \chi_\epsilon^{-\nu} - M$. It is apparent from Fig. 1 that we must distinguish two different cases:

(a) If $M < M^*$ (region $A$ in Fig. 1), the term between the brackets in (2) is positive, and the resulting equation is very similar to (1) describing the extent of the stabilized zone in SD. As in that case, it admits a solution for all values of $Ca$, with the same dependence on parameters as $\sigma$. Moreover, and for the same reasons as in SD, $p$ will be decreasing with distance for all $\chi(t) \leq \chi_\epsilon$, and the problem is IPSG. Thus, when $M < M^*$ the displacement is an unconditionally stabilized displacement. This conclusion is similar to that of conventional stability analyses [5] (where $M^* = 1$ in Chuoke et al., $M^* > 1$ and also dependent on other petrophysical properties in Yortsos and Hickernell), but is reached here using IPG.

(b) If $M > M^*$, the solution $\chi_\epsilon$ can lie either in region $B_d$ or in region $B_s$ (Fig. 1). If in region $B_d$, the term in the brackets of (2) is negative, which means that $p$ increases with distance. In this case, therefore, the pattern when $\chi_\epsilon$ is reached will depart from percolation toward the CVF regime. Now, $\chi_\epsilon$ solves

$$-\chi_\epsilon^{-\nu} \left( C_1 \chi_\epsilon^{-\nu} - M \right) \sim \frac{C_1}{Ca} \epsilon$$

(3)

However, a solution of (3) does not exist for all $M$ or $Ca$. Indeed, the LHS of this equation goes through a maximum, which, when substituted back in (3) gives the following condition for a solution to exist

$$\frac{C_1}{Ca} M^{\frac{\zeta + \nu(D-2)}{\zeta + \nu(D-2)}} \geq O(\epsilon)$$

(4)
Figure 1: The ratio $\frac{M\Delta P_{nw}}{\Delta P_w}$ (first term between brackets in (2)) vs. the extent $\chi_e$. 
Figure 2: Phase diagram of fully developed displacement in drainage.
where we grouped all constants into an $O(1)$ parameter in the RHS. This inequality expresses the condition for the existence of the CVF regime. It shows that for $M$ above $M^*$, the displacement will become CVF provided that $Ca$ or $M$ are sufficiently large. Otherwise, the pattern will remain at percolation as the displacement proceeds throughout region $B_d$, as well as after $B_d$ is exited (at $\chi_0$, Fig. 1) and region $B_s$ is entered. In the latter region, the term within the brackets in (2) is positive, the resulting equation having a solution for all values of $M$ or $Ca$. Reasoning as in case (a), we conclude that if (4) is violated, the transition will be toward a SD.

These results can be portrayed in a phase diagram of fully developed drainage (Fig. 2). The two regimes SD and CF are separated by a line, which at large $M$ is asymptotically straight with slope $\frac{\zeta+1+\nu(D-1)}{\zeta+\nu(D-2)}$, and as $M$ approaches $M^* \sim 1$ becomes asymptotically vertical. By taking a qualitative analogue of (2) using the simpler quadratic equation $|c\chi^2 - M\chi| = a$, it can be shown that when the boundary is crossed, there is a discontinuity indicative of a first-order phase transition. This interesting behavior is also a result of the application of IPG to this problem and needs to be explored further.

4 Conclusions

We summarize the above as follows: A stabilized displacement (and a continuum description) is possible either if $M < M^*$ for all $Ca$, or if $M > M^*$ but for sufficiently small $Ca$, as dictated by (4). Otherwise, the displacement will be destabilized. The curve separating the two regions also serves to delineate the validity of the conventional continuum description, which only applies to a SD displacement. The resolution of this has been a long standing problem.

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6 References