On elastoplastic analysis of 1-D structural member subject to combined bending and torsion

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Abstract

A new model of an elastoplastic rectangular beam element is presented. The model is used in an elastoplastic analysis of the beam due to combined bending and torsional moment by introducing a plastic-depth factor. Using the plastic node method, the yielding process of a section can be included in the analysis without modeling the beam as a plane or a solid structural member. Therefore, the computational cost on an elastoplastic analysis is reduced. Examples shows that the present analysis gives a more realistic and conservative collapse load.

1. Introduction

In an elastic-plastic analysis, the yielding in a beam element is usually assumed to be controlled only by bending. However, for member that also carries torsional moment such as in grillage structures, the interaction between bending and torsion at yielding element should be considered.

The plastification in a beam member is often assumed to lump at a plastic hinge section which has a zero length while the material is assumed to be elastic perfectly-plastic. The yielding initiates on edges of a section when the bending and the torsion satisfy the fully-plastic condition, i.e. the analysis involves only the elastic and the fully-plastic section. A collapse load obtained by these assumptions is expected to be higher than the load obtained by considering the plastification process of the section.
An elastoplastic analysis of a beam element is presented by the author which includes the interaction between bending and torsion during the plastification at the plastic hinge. To accommodate the plastification of layers at the yielding section, an elastoplastic beam model is introduced so that the beam does not need any mesh or layer refinement.

2. Plasticity and Plastic Node Method

Yielding is assumed at any point of a beam element due to combined normal and shear stresses when the von Mises yield condition is satisfied:

$$\sigma^2 + 3 \tau^2 = \sigma_y^2$$  \hspace{1cm} (1)

where $\sigma_y$ is the normal yield stress of the material due to axial tension. This condition at a fully-plastic beam section is satisfied when the bending moment, $M$, and torsional moment, $T$, at that section satisfy the yield interaction [1]:

$$\left( \frac{M}{M_p} \right)^2 + \left( \frac{T}{T_p} \right)^2 = 1$$  \hspace{1cm} (2)

where $M_p$ is the fully-plastic bending moment of the section in pure bending and $T_p$ is the fully-plastic torsional moment of the section in pure torsion. The yield function, $f'$, can be written as

$$f' = \left( \frac{M}{M_p} \right)^2 + \left( \frac{T}{T_p} \right)^2 - 1$$  \hspace{1cm} (3)

where $f' = 0$ refers to a yield condition.

The Plastic Node Method was introduced by Ueda [2] for an elastic-plastic analysis of structure by assuming the plasticity is lumped as a fully-plastic section at beam-ends (i.e. plastic node).

The elastic force-displacement relation can be written as:

$$k \ u = r$$  \hspace{1cm} (4)

where $u$ is the displacement vector of the beam element, $k$ is the beam stiffness matrix, and $r$ is the force vector. The displacement increment, $\delta u$, can be written as

$$\{ \delta u \} = \{ \delta u^e \} + \{ \delta u^p \} = \left\{ \frac{\delta u^e}{\delta u^e} \right\} + \left\{ \frac{\delta u^p}{\delta u^p} \right\}$$  \hspace{1cm} (5)
where subscripts $i$ and $j$ refer to two beam-ends, superscript $e$ and $p$ refer to the elastic and the plastic part of the displacement. For the associated flow rule the plastic displacement increment is expressed as

$$\{ \delta u^p \} = [\lambda] \left\{ \frac{\partial f}{\partial r} \right\}$$

$$\left\{ \frac{\partial f}{\partial r} \right\} = \left\{ \Psi_i \right\} = \left\{ \begin{array}{c} \frac{\partial f}{\partial r_i} \\ \frac{\partial f}{\partial r_j} \end{array} \right\}$$

and $[\lambda]$ is the scalar factor of proportionality matrix or plastic multiplier matrix. The increment of forces can be expressed as

$$\{ \delta r \} = \left\{ \begin{array}{c} \delta r_i \\ \delta r_j \end{array} \right\} = \left[ \begin{array}{cc} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{array} \right] \left\{ \begin{array}{c} \delta u_i^p \\ \delta u_j^p \end{array} \right\} = \left[ \begin{array}{cc} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{array} \right] \left\{ \begin{array}{c} \delta u_i - \delta u_i^p \\ \delta u_j - \delta u_j^p \end{array} \right\}$$

$$\left[ \begin{array}{cc} k_{ii} & k_{ij} \\ k_{ji} & k_{jj} \end{array} \right] \left\{ \begin{array}{c} \delta u_i - \lambda_i \Psi_i \\ \delta u_j - \lambda_j \Psi_j \end{array} \right\}$$

$$\{ \delta f \} = \left\{ \begin{array}{c} \delta f_i \\ \delta f_j \end{array} \right\} = \left[ \begin{array}{cc} \Psi_i & 0 \\ 0 & \Psi_j \end{array} \right] \left\{ \begin{array}{c} \delta r_i \\ \delta r_j \end{array} \right\}$$

where the zero value of $\delta f_i$ or $\delta f_j$ is assumed if node $i$ or $j$ is plastic.

For a plastic node of a standard beam element where $r = [Q_k, M_k, T_k]^T$, $\Psi_i^T$ and $\Psi_j^T$ can be written in the form of

$$\Psi_k^T = \left\{ \frac{\partial f}{\partial r_k} \right\}^T = \left[ \begin{array}{c} 0 \\ 2 \frac{M_k}{M_p^2} \\ 2 \frac{T_k}{T_p^2} \end{array} \right]$$

where subscript $k$ refers to the plastic node $i$ or $j$.

The two equations, Equation 8 and Equation 9, can be used to construct the force-displacement relation for the 1-D beam element,

$$\{ \delta r \} = \left\{ \begin{array}{c} \delta r_i \\ \delta r_j \end{array} \right\} = \left[ \begin{array}{cc} k_{ii} & k_{ij}^{ep} \\ k_{ji}^{ep} & k_{jj}^{ep} \end{array} \right] \left\{ \begin{array}{c} \delta u_i \\ \delta u_j \end{array} \right\}$$

where superscript $ep$ indicates the elastic-plastic stiffness matrix.
As illustrations, let consider two cases of plasticity of a beam element.

Case 1: Node $i$ is fully-plastic, node $j$ is still elastic.
In this case, $\delta f_i = 0$ and $\lambda_j = 0$, and the elastic-plastic stiffness matrix becomes

$$k^{op} = \begin{bmatrix} k_{ii} - \mu_1 k_{ii} \Psi_i^T \Psi_i k_{ii} & k_{ij} - \mu_1 k_{ii} \Psi_i \Psi_j^T k_{ij} \\ k_{ji} - \mu_1 k_{jj} \Psi_j \Psi_i^T k_{ij} & k_{jj} - \mu_1 k_{jj} \Psi_j \Psi_j^T k_{jj} \end{bmatrix}$$  \hspace{1cm} (12)

where:

$$\mu_i = \frac{1}{\Psi_i^T k_{ii} \Psi_i}$$  \hspace{1cm} (13)

or

$$k^{op} = k - \Omega_i$$  \hspace{1cm} (14)

where:

$$\Omega_i = \begin{bmatrix} \mu_i k_{ii} \Psi_i \Psi_i^T k_{ii} & \mu_i k_{ii} \Psi_i \Psi_j^T k_{ij} \\ \mu_i k_{jj} \Psi_j \Psi_i^T k_{ij} & \mu_i k_{jj} \Psi_j \Psi_j^T k_{jj} \end{bmatrix}$$  \hspace{1cm} (15)

The matrix $\Omega_i$ describes the stiffness matrix decrease due to the fully-plastic section (plastic node) which is assumed at end-node $i$.

Case 2: Node $i$ is still elastic, node $j$ is fully-plastic.
In this case, $\delta f_i = 0$ and $\lambda_i = 0$. The elastic-plastic stiffness matrix becomes

$$k^{op} = k - \Omega_j$$  \hspace{1cm} (16)

where:

$$\Omega_j = \begin{bmatrix} \mu_2 k_{ii} \Psi_i \Psi_i^T k_{ii} & \mu_2 k_{ii} \Psi_i \Psi_j^T k_{ij} \\ \mu_2 k_{jj} \Psi_j \Psi_i^T k_{ij} & \mu_2 k_{jj} \Psi_j \Psi_j^T k_{jj} \end{bmatrix}$$  \hspace{1cm} (17)

and

$$\mu_2 = \frac{J}{\Psi_j^T k_{jj} \Psi_j}$$  \hspace{1cm} (18)

3. Elastoplastic Analysis

The plastification of a section due to combined bending and torsion is different from the plastification in pure bending or in pure torsion individually. The complexity in the analysis may arise if the analysis must include the yielding of the material along the length, the width, and the thickness of the beam element. Also, the ratio of bending to torsional moment will vary during the plastification.
An elastoplastic section that models the plastification of a rectangular beam section is introduced as shown in Figure 1. This model accommodates the yielding of the section via a plastic depth factor, $\alpha$, which describes how far the plastification has occurred from both edges of the section toward the center of the section, due to combined bending and torsion. At an elastoplastic section, the normal stress, $\sigma^*$ and the shear stress, $\tau^*$ acting on the yielding region must satisfy the yield condition in Equation 1.

$$A = A$$

$$A = \frac{h}{\alpha b}$$

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(a) (b)

Figure 1: A Rectangular beam section model.
   a. Elastic Section   b. Elastoplastic section

The bending moment, $M$, acting on the section can be written as a summation of $M_1$, $M_2$ and $M_3$, as shown in Figure 2:

$$M_1 = (h - \alpha b)\left(\alpha \frac{b}{2}\right) b \sigma^*$$  \hspace{1cm} (19)$$

$$M_2 = 2 \left\{ \frac{1}{4} \left(\alpha \frac{b}{2}\right) (h - \alpha b)^2 \sigma^* \right\}$$  \hspace{1cm} (20)$$

$$M_3 = \frac{b}{6} h (h - \alpha b)^2 (1 - \alpha) \sigma^*$$  \hspace{1cm} (21)$$

Although the model does not satisfy the compatibility on the yielding section, nevertheless the bending moment contribution from normal stress near the neutral axis is relatively small compared with the one from normal stress acting near the bottom and top layer of the section.
Figure 2: Bending moment, $M$, acting on the elastoplastic section model.

$$M = M_1 + M_2 + M_3$$

The torsional moment, $T$, can be written as a summation of $T_1$, $T_2$, and $T_3$:

$$T_i = 4 \left\{ \frac{1}{2} \left( \frac{a}{b} \right)^2 \cdot \frac{a}{d} \cdot \tau^* \right\}$$

$$+ 2 \left\{ (h - a) \cdot \frac{a}{b} \cdot \frac{a}{d} \cdot \tau^* \right\}$$

$$+ 2 \left\{ (h - a) \cdot \frac{a}{b} \cdot \frac{a}{d} \cdot \tau^* \right\}$$

$$= \frac{1}{2} a^2 b^2 (b + h - \frac{a}{d}) \tau^*$$

(22)
\[ T_2 = 2 (h - \alpha b) (b - \alpha b) \left( \frac{\alpha b}{2} \right) \tau^* \]  
(23)

\[ T_3 = c_s (h - \alpha b) (b - \alpha b)^2 \tau^* \]  
(24)

where \( T_1 \) and \( T_2 \) are equal to two times the volume of the solid bodies shown in Figure 3, and \( c_s \) can be approximate by the Roarks's formula [3] :

\[ c_s \equiv \frac{h}{l} \left[ 1 - 0.6095 \left( \frac{h}{l} \right)^2 + 0.0065 \left( \frac{h}{l} \right)^3 - 1.8063 \left( \frac{h}{l} \right)^3 + 0.9100 \left( \frac{h}{l} \right)^4 \right] \]  
(25)

Figure 3: Torsional moment, \( T \), acting on the elastoplastic section model

\[ T = T_1 + T_2 + T_3 \]
The bending and the torsional moment at a yielding section can be written as

\[ M = M_1 + M_2 + M_3 \]
\[ = \frac{1}{6} bh^2 \sigma^* \cdot \theta_m \]  
\[ T = T_1 + T_2 + T_3 \]
\[ = bh^2 \tau^* \cdot \theta_t \]  
where:
\[ \theta_m = 1 + \frac{1}{2} \alpha \left[ 1 + (1 - \alpha) \left( \frac{b}{h} - \alpha \left( \frac{b}{h} \right)^2 \right) \right] \]
\[ \theta_t = (1 - \alpha \frac{b}{h})(1 - \alpha) \left\{ \alpha + c, (1 - \alpha) \right\} + \frac{1}{2} \alpha^2 \left\{ 1 + (1 - \frac{4}{3} \alpha) \frac{b}{h} \right\} \]  

When the edges begin to yield, \( \alpha = 0, \theta_m = 1 \) and \( \theta_t = c_s \), while for a fully-plastic section, \( \alpha = 1, \theta_m = 1.5 \) and \( \theta_t = (3 - b/h)/6 \). These values agree with the analytical values for elastic and fully plastic section.

Substituting the values of \( \sigma^* \) and \( \tau^* \) in Equation 26 and Equation 27 into Equation 1 will give a new yield criteria for an elastoplastic section:

\[ \left( \frac{L_5}{\theta_m} \right)^2 \left( \frac{M}{M_p} \right)^2 + \left( \frac{3 \cdot \frac{b}{h}}{6 \theta_t} \right)^2 \left( \frac{T}{T_p} \right)^2 = 1 \]  

The plastic-depth factor, \( \alpha \), can be computed numerically for any applied bending and torsional moment on an elastoplastic rectangular section, by using Equation 38 and Equation 39.

The stiffness matrix of a yielding member decreases. To obtain a conservative collapse load, the values of moment of inertia of yielding members are approximated by \( I_p = \omega I \) and \( J_p = \omega J \), where \( \omega = (l-\alpha)^t \). Thus, the elastic-plastic stiffness matrix of the beam element, \( K^{ep} \), is then computed as

\[ K^{ep} = K - (1 - \omega) \Omega \]  

Once a section becomes fully-plastic, \( \omega = 0 \), and Equation 31 becomes Equation 14 or Equation 16.
4. Some Examples

a. A right angle bent structure due to a concentrated vertical load shown in Figure 4 was analysed by Ueda [2], Heyman [4], and Hodge [5] to obtain the collapse load by assuming the elastic and the fully-plastic section. The beam properties are \( E = 3 \times 10^7 \) psi, \( G = 1.2 \times 10^7 \) psi, \( \sigma_y = 20,000 \) psi, \( b = h = 0.75 \) in., \( L = 20 \) in. Previous analysis gives a collapse load \( P = 499 \) lbs when the vertical deflection at node B is about 1.0 in. The present elastoplastic analysis gives \( P = 480 \) lbs with the load-displacement curve shown in Figure 5. The first yielding occurs at node D when \( P = 220 \) lbs. In present analysis the variation of bending and torsional moment is non-linear with a smooth transition from elastic to elastoplastic state (Figure 6). After the edges of the section yields at \( P = 220 \) lbs, the rate of change of the bending moment decreases while the torsional moment increases faster than in elastic state. Figure 7 shows the change of \( (M/M_p)^2 + (T/T_p)^2 \) and the plastic-depth factor, \( \alpha \), of the section in node D.

b. An inelastic lateral buckling of intersecting connected beams shows in Figure 8 was analysed using the present analysis. The beam properties are: \( E = 3 \times 10^7 \) psi, \( G = 1.16 \times 10^7 \) psi, \( \sigma_y = 20,000 \) psi, \( b = 0.125 \) in, \( h = 5 \) in., \( l_1 = 20 \) in, \( l_2 = 10 \) in. All supports are fixed. The stability analysis by Morchi [6] gives an elastic buckling load \( P = 5,020 \) lbs, while the virtual work method gives an upper bound collapse load \( P = 4.688 \) lbs. By including the stability matrix in the analysis, the present elastoplastic analysis gives an inelastic buckling load \( P = 3,925 \) lbs as shown in Figure 9.

5. Conclusion

A model of plastification of a rectangular beam element has been presented. The model can be used to perform an elastoplastic analysis of structure with beam member where the yielding of the beam section is governed not only by the bending but also by the torsion at the beam-ends. The inelastic buckling can also be detected using the analysis. The present analysis gives a more realistic and conservative collapse load. Since the beam element is analysed as a 1-D element and not by a plane element or a solid element to include the yielding process at a beam section, the analysis could save considerably large computational cost.
Figure 4: A Right angle bent structure under a vertical load

\[ P \times 10 \text{ lbs} \]

![Graph showing load-displacement relation with deflection at Node B in inches.]

- Node C yields \( P = 426 \text{ lbs} \)
- Node A of AB yields \( P = 370 \text{ lbs} \)
- Node D of CD yields \( P = 220 \text{ lbs} \)

Figure 5: Load-displacement relation of structure in Figure 4
Figure 6: Bending and torsional moments at Node D in Figure 4. Dash lines corresponds to previous analysis.

Figure 7: The variation of plastic depth factor, $\alpha$, on yielding section at node D.
Figure 8: A Rigidly connected cross-beams.

Figure 9: Load-displacement relation of structure in Figure 8
References