A new Crank-Nicholson algorithm for solving the diffusive wave flood routing equation along a complex channel network

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Abstract

The diffusive wave equation is generally used in flood routing in rivers. We developed a modified form of this equation and proposed a new finite-differences resolution algorithm which is better adapted to flood routing along a complex river network. The new algorithm solve the diffusive wave equation in time then in space, opposed to the conventional approach of in space and then progressing in time. The accuracy of our new algorithm was compared to a traditional one by numerical experimentation and then applied to flood routing simulation over the Gardon d'Anduze catchment in South France.

1 Introduction

The dynamic modeling of a one-dimensional unsteady flow in open channels is usually based on the numerical solution of the Saint-Venant equations. The solution of the Saint-Venant system, which constitutes a system of hyperbolic quasi-linear partial differential equations, has given rise to a number of numerical methods because no analytical solutions were available. When large systems are considered, it takes time to solve the overall problem. Hydrologists have proposed extremely simplified models as alternative: storage routing, Muskingum routing, etc. All of them require little computer time, but can only describe the overall response of a river reach, and do not simulate the hydraulic structures inserted within the reach. This is the reasons why many hydrologists have tried to simplify the Saint-Venant equations.

In most of the practical applications, the acceleration terms in the Saint-Venant equations can be neglected, and the system is reduced to only one parabolic equation: the diffusive wave equation. Under some conditions, this equation may have an analytical solution (Hayami [1]). Methods based upon the finite-difference discretization techniques are generally used to calculate discharges at each time step (Richtmeyer and Morton [2]; Remson et al. [3]). We propose a modified form of the diffusive wave equation and a new
resolution method, well adapted to flood routing along a complex river network. The accuracy of this new method is compared to a traditional scheme by numerical experimentation and then tested for flood routing over the Gardon d'Anduze catchment.

2 The flood routing problem

In 1871, Barré de Saint-Venant formulated the classical system of partial differential equations that allow study of one-dimensional, gradually varied unsteady flow. The two equations, describing the mass balance and the energy balance in a reach, can be written as follows

\[
\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0 \tag{1}
\]

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x} \left( \frac{Q^2}{A} \right) + gA \left( \frac{\partial z}{\partial x} + j \right) = 0 \tag{2}
\]

where \( Q(x,t) \) is the discharge, \( z(x,t) \) the water profile elevation above an horizontal datum, \( A(z,x) \) the cross-sectional area of the flow, \( j \) the friction losses slope, \( x \) the downstream distance and \( t \) the time.

The methods based upon the finite-difference discretization techniques are the most widely used (Cunge et al., [4]). Applications of finite-elements schemes were also published, although finite-elements techniques are not particularly recommended for the integration of hyperbolic systems and do not offer great advantages over the finite-difference schemes for the solution of one dimensional problems. In any case, the computer time required for the solution of large systems is significant. There is also a considerable difference in data requirements for the full Saint-Venant equations and for simplified forms like the diffusive wave equation. The latter requiring considerably less resources. This is the reasons why many hydrologists have tried to simplify the Saint-Venant system.

In most practical applications, the acceleration terms (terms containing the derivative of \( Q \) with respect to \( x \) and \( t \)) in the momentum balance equation can be neglected since they are small in comparison to the channel bed slope. By combining the two equations (1) and (2), we obtain the diffusive wave model

\[
\frac{\partial Q}{\partial t} = -C \frac{\partial Q}{\partial x} + D \frac{\partial^2 Q}{\partial x^2} \tag{3}
\]

where \( C \) and \( D \) are non-linear functions of \( Q \) and \( z \) and are generally known as celerity and diffusivity, respectively. For this problem the boundary conditions are \( Q(x,0) \) and the upstream inflow \( Q(0,t) \). In general, when \( C \) and \( D \) are functions of \( x \) and \( t \) and when the lateral inflow is important, finite-difference schemes are still required but if the coefficients \( C \) and \( D \) can be assumed constant, the diffusive wave equation may have an analytical solution : The Hayami model [1]

\[
Q(x,t) = Q(0,0) + \frac{L}{2(\pi D)^{\frac{1}{2}}} \cdot \exp \frac{CL}{4D} \int_0^t (Q(0,t-\tau) - Q(0,0)) \cdot \exp \frac{-CL(L + C\tau)}{4D\tau^{\frac{3}{2}}} d\tau \tag{4}
\]

\( L \) being the total length of the river reaches.
3 The Crank-Nicholson scheme over x (CNX)

As a first step in obtaining a finite-difference solution to (3), a grid is superimposed on the region of interest in the xOt plane as shown in Figure 1.a. \( \Delta x \) and \( \Delta t \) are respectively the space and time steps of discretization. Let the nodes be numbered so that their coordinates are given by \((x_i, t_j)\), where \( x_i = i \Delta x \) with \( 0 \leq i \leq n \) and \( t_j = j \Delta t \) with \( 0 \leq j \leq m \). Finally denote \( Q(x_i, t_j) \) by \( Q(i,j) \).

![Figure 1. a. The discretization of Crank-Nicholson scheme over x (CNX). b. The discretization of Crank-Nicholson scheme over t (CNT).](image)

Richtmeyer and Morton [2] tested a variety of finite-differences systems for the diffusive wave equation and preferred the implicit Crank-Nicholson [5] scheme when dealing with smooth functions because its truncation error is smaller and is unconditionally stable. The linearization of the space second derivative in equation (3) needs at least three nodes over the x-axis. Around a typical central node \( P \) in the grid system, the scheme uses six nodes. To write the difference equation, we introduce the abbreviations

\[
\begin{align*}
  u_{i-1} &= Q(i-1,j) \\
  u_i &= Q(i,j) \\
  u_{i+1} &= Q(i+1,j) \\
  v_{j-1} &= Q(i,j-1) \\
  v_j &= Q(i,j-1) \\
  v_{j+1} &= Q(i,j+1) \\
\end{align*}
\]

The points shown are those points of the grid in the xOt plane used in one application of the formula. Using the Taylor series truncated, the problem may be represented at the point \( P \) as

\[
\frac{\partial Q}{\partial t} = \frac{u_i - v_i}{\Delta t} \\
\frac{\partial^2 Q}{\partial x^2} = \frac{1}{2} \left( \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + \frac{v_{j+1} - 2v_j + v_{j-1}}{\Delta x^2} \right)
\]

Replacing the terms (7) and (8) in (3) gives

\[
p_i, u_{i-1} + q_i, u_i + r_i, u_{i+1} = p_i, v_{i-1} + q_i, v_i + r_i, v_{i+1}
\]

with \( p_i = -\frac{h}{4} + \frac{g}{2} \), \( q_i = 1 + g \), \( r_i = \frac{h}{4} + \frac{g}{2} \), \( p_i = \frac{h}{4} + \frac{g}{2} \), \( q_i = 1 - g \), \( r_i = -\frac{h}{4} + \frac{g}{2} \)

There are as many equations as interior nodes. These equations must be solved subject to boundary conditions of some sort at say i=0, i=n and j=0. The
linear system is a very special one, because all the elements of the corresponding matrix vanish except those on three diagonals. Any of the standard methods for solving a linear system could be used but to be efficient, a method of solution should take advantage of this sparseness (Smith [6]). Then the resolution method calculates discharge at each space step over the x-axis. We call this method "CNX". In practical applications, this method allows use of variable C(Q) and D(Q), but needs regular space steps.

4 Modified diffusive wave equation (CNT)

We propose to modify the diffusive wave equation so that variable space steps can be used in the numerical solution by Crank-Nicholson. This new scheme is better adapted to flood routing in a river network and allows to take into account the hydraulic characteristics C(Q) and D(Q) of each reach. The new diffusive wave equation form exchanges the role of x and t axes (Figure 1.b). The space second derivative in equation (3) is replaced with time second derivative. A similar Crank-Nicholson approximation is written, and a hydrograph is calculated at each space step \( x=0, x=\Delta x, x=2.\Delta x, \ldots, x=n.\Delta x \) or with variable space steps. We call this method "CNT". Because \( C \neq 0 \) the diffusive wave equation (3) can be written

\[
\frac{\partial Q}{\partial x} = -\frac{1}{C} \frac{\partial Q}{\partial t} + \frac{D}{C} \frac{\partial^2 Q}{\partial x^2},
\]

If one assumes that \( C \) and \( D \) are constant, differentiating equation (12) gives

\[
\frac{\partial^2 Q}{\partial x \partial t} = -\frac{1}{C} \frac{\partial^2 Q}{\partial t^2} + \frac{D}{C} \frac{\partial^3 Q}{\partial x^3}.
\]

By combining the two equations (13), we obtain

\[
\frac{\partial^2 Q}{\partial x^2} = \frac{1}{C^2} \frac{\partial^2 Q}{\partial t^2} + \frac{D}{C^2} \frac{\partial^3 Q}{\partial x^2 \partial t} + \frac{D}{C} \frac{\partial^3 Q}{\partial x^3}.
\]

Substituting (14) into the equation (3) gives

\[
\frac{\partial Q}{\partial t} = -\frac{C}{C^2} \frac{\partial Q}{\partial x} + \frac{D}{C} \frac{\partial^2 Q}{\partial x^2} - \frac{D^2}{C^2} \frac{\partial^3 Q}{\partial x^2 \partial t} + \frac{D^2}{C} \frac{\partial^3 Q}{\partial x^3}.
\]

The space second and third derivatives in equation (15) must be replaced with time second derivative. If higher order derivative terms are neglected after derivation of equations (13), we obtain

\[
\frac{\partial^3 Q}{\partial x^2 \partial t} \approx -\frac{1}{C} \frac{\partial^3 Q}{\partial x \partial t^2} + \frac{D}{C} \frac{\partial^3 Q}{\partial x^3}.
\]

We can subsequently modify the diffusive wave equation (15) by substituting (16)

\[
\frac{\partial Q}{\partial x} = -\frac{1}{C} \frac{\partial Q}{\partial t} + \frac{D}{C^3} \frac{\partial^2 Q}{\partial t^2} + \frac{2D^2}{C^2} \frac{\partial^3 Q}{\partial x \partial t^2}.
\]

We then obtain a new modified form of the diffusive wave equation where the space second derivative is replaced with time second derivative. Because higher order derivations were neglected, the modified form equation is not strictly equivalent to the original equation even for constant \( C \) and \( D \). A similar implicit Crank-Nicholson scheme with six nodes is used to solve equation (17).
Using the Taylor series truncated and Crank-Nicholson [5] approximation as in equations (7) and (8) and by exchanging the role of "i" and "j", the problem may be represented at a point P as

\[ p_j \cdot u_{j-1} + q_j \cdot u_j + r_j \cdot u_{j+1} = p_j \cdot v_{j-1} + q_j \cdot v_j + r_j \cdot v_{j+1} \]  

(20)

with

\[ p_j = \frac{h}{4} - \frac{g}{2} - 2k \]
\[ q_j = 1 + g + 4k \]
\[ r_j = \frac{h}{4} + \frac{g}{2} - 2k \]

(21)

\[ p_j = \frac{h}{4} + \frac{g}{2} - 2k \]
\[ q_j = 1 - g + 4k \]
\[ r_j = \frac{h}{4} - \frac{g}{2} - 2k \]

(22)

\[ h = \frac{\Delta x}{C \cdot \Delta t} \]
\[ g = \frac{D \cdot \Delta x}{C^3 \cdot \Delta t^2} \]
\[ k = \frac{g^2}{h^2} = \frac{D^2}{C^4 \cdot \Delta t^2} \]

(23)

We obtain a linear system similar to that in equation (9). A new term \( k \) appears in (21) and (22). The CNT method has the same properties as the CNX method but also enables to use variable space steps and so is well adapted to flood routing along a complex river network. This method takes into account lateral inflow hydrograph which could be added at each space step to the hydrograph.

5 Analysis of errors due to numerical algorithms

The accuracy of a numerical method depends on the boundary conditions (particularly the form of inflow hydrograph), the functions \( C(Q) \) and \( D(Q) \), the finite-difference algorithm and the space and time steps of discretization. We analyze the problem in nondimensionalized space that allows the decrease in the number of variables. A test problem was designed to show the performance of the algorithms (CNX and CNT) for different boundary conditions. A schematic inflow is used as the boundary condition at \( x=0 \) with

\[ Q(0, t) = Q_0 \cdot \exp \left( -\frac{w^2}{t} \right) \]

where \( Q_0 \) is a reference discharge, \( w \) a time that represents the centre of gravity of the hydrograph and \( z \) a form parameter. Then, it is convenient to nondimensionalize this problem with the substitutions given in Table 1 supposing \( C \) and \( D \) are constants.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Nondimensionalization</th>
</tr>
</thead>
<tbody>
<tr>
<td>Space : x and time : t</td>
<td>( x_* = C \cdot x / D ) and ( t_* = C^2 \cdot t / D )</td>
</tr>
<tr>
<td>Inflow characteristics: w, z and ( Q_0 )</td>
<td>( w_* = C^2 \cdot w / D ); ( z_* = z ) and ( Q_* = Q_0 / Q_0 )</td>
</tr>
<tr>
<td>Space step : ( \Delta x ) and time step : ( \Delta t )</td>
<td>( \Delta x_* = C \cdot \Delta x / D ) and ( \Delta t_* = C^2 \cdot \Delta t / D )</td>
</tr>
<tr>
<td>Total length of reaches : ( L )</td>
<td>( L_* = C \cdot L / D )</td>
</tr>
</tbody>
</table>

Table 1. List of Nondimensionalized Variables

These transformations reduce the equation (3) to
For this problem the upstream boundary condition is

\[ Q_+ (0, t_+) = \exp \left( \frac{t_+}{w_+} \right) \]

and the assumption of an infinite channel as downstream boundary condition.

The two algorithms (CNX and CNT) were compared to the exact solution given by the analytical method of Hayami (eqn. (4)) with a small time step of numerical integration. The three finite-differences algorithms were compared with different values of \((w^+, z^+)\) with \(L^+ = w^+\) and the space and time steps \(\Delta x^+ = \Delta t^+ = w^+/100\) corresponding to Courant numbers. Figure 2 shows an example of an inflow hydrograph, the outflow calculated with Hayami analytical method and the routing outflow calculated by the numerical method and Table 2 shows the three error criteria by which the two outflows were compared.

![Figure 2. Comparison of outflow hydrographs between the exact analytical method of Hayami and a finite-difference numerical method.](image)

<table>
<thead>
<tr>
<th>Calculated Variables</th>
<th>Nondimensional Error Criteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maximum discharge amplitude : a</td>
<td>eQ = b/a</td>
</tr>
<tr>
<td>Difference between maximum : b</td>
<td>eT = f/e</td>
</tr>
<tr>
<td>Oscillation : d</td>
<td>Osc = d/a</td>
</tr>
<tr>
<td>Time interval between maximum : e</td>
<td></td>
</tr>
<tr>
<td>Difference in time position : f</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. List of Calculated Variables and Error Criteria.

The test problem compared the two algorithms (CNX and CNT) for different inflow hydrographs as it is shown in Table 3. It is normal that the CNX algorithm offers superior accuracy over the widest range of conditions because the finite-difference method was compared to the analytical solution of equation (3) and not the analytical solution of the modified form with truncated terms of the diffusive wave equation (17).
At great wave lengths \( w^+ = 200 \) the two algorithms are very efficient and give results comparable to the exact solution. Note that for short wave lengths \( w^+ < 50 \) or \( z^+ > 50 \), results are also sensitive to space and time steps and small discretization steps must be used to take into account the form of input hydrograph. In practice, the accuracy of a method depends also on the accuracy of the inputs: the quality of input hydrographs, the structure and the geometry of channel network, the amplitude and the distribution of lateral inflow, the space and time steps and the functions \( C(Q) \) and \( D(Q) \). Generally, the error due to numerical method could be neglected since it is small (less then 1 to 3%) in comparison with the accuracy on the maximum discharge measured (10 to 25%) or the Manning coefficient of roughness (10 to 25%) used to calculate \( C(Q) \). In comparison with the CNX algorithm, the new algorithm CNT enable to use variable space steps as well as any distribution in space and time of lateral inflow and so is well adapted to flood routing along complex channel network extracted from Digital Elevation Model.

### 6 Applications

The Garden d’Anduze catchment (542 km²) in South France has been selected to test the proposed CNT method. The catchment is equiped with seven rain gauges and a streamflow recorder at Anduze. The catchment was divided into 80 sub-catchments by analyzing Digital Elevation Models (Figure 3.a). Each sub-catchment has physiographic characteristics of area, elevation range, mainstream length and mean slope, and is connected to a reach of the channel network. The distribution of rainfall losses during a storm was made using an empirical Philip-type infiltration equation

\[
f = \frac{S}{2} \sqrt{t + A}
\]

where \( f \) is the infiltration rate at time \( t \) and \( S \) and \( A \) are two parameters reflecting the properties of soil. \( S \) and \( A \) were determined by trial and error such that the calculated effective rainfall was equal to the surface runoff.

The runoff from each sub-catchment constitutes lateral inflow into each reach. For each reach, hydrographs are added at the upstream then routed through the channel network to the downstream outlet to give the full hydrograph. The CNT method was used with a mean values of the celerity is 1.9 m s⁻¹, diffusivity 2400 m² s⁻¹, time step \( \Delta t = 15 \) minutes and space step \( \Delta x = 1000 \) m. Figure 3.b shows for a flood event, the rainfall and the observed and the calculated discharges.

### Table 3. Comparison of the accuracy criteria for CNX and CNT.

<table>
<thead>
<tr>
<th>( w^+ )</th>
<th>( L^+ )</th>
<th>( z^+ )</th>
<th>( \Delta x^+ )</th>
<th>( \Delta t^+ )</th>
<th>( eQ % ) CNX</th>
<th>( eQ % ) CNT</th>
<th>( eT % ) CNX</th>
<th>( eT % ) CNT</th>
<th>Osc% CNX</th>
<th>Osc% CNT</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>200</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>-0.04</td>
<td>-0.13</td>
<td>0.98</td>
<td>0.98</td>
<td>0</td>
<td>-0.01</td>
</tr>
<tr>
<td>200</td>
<td>200</td>
<td>5</td>
<td>2</td>
<td>2</td>
<td>-0.04</td>
<td>-0.09</td>
<td>1.49</td>
<td>0.99</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>200</td>
<td>200</td>
<td>50</td>
<td>2</td>
<td>2</td>
<td>-0.06</td>
<td>-0.30</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-0.01</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>1</td>
<td>0.5</td>
<td>0.5</td>
<td>-0.49</td>
<td>-0.36</td>
<td>0.94</td>
<td>0.94</td>
<td>0</td>
<td>-0.24</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>5</td>
<td>0.5</td>
<td>0.5</td>
<td>-0.06</td>
<td>-0.54</td>
<td>0.98</td>
<td>0.98</td>
<td>0</td>
<td>-0.09</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>50</td>
<td>0.5</td>
<td>0.5</td>
<td>-0.14</td>
<td>-2.61</td>
<td>1.05</td>
<td>1.05</td>
<td>0</td>
<td>-0.55</td>
</tr>
</tbody>
</table>
Figure 3.  
(a) Subdivision of the basin into sub-catchments.  
(b) Flood routing over the Gardon d'Anduze river.

7 Conclusions

A modified form of the diffusive wave equation and a new resolution method (CNT), well adapted to flood routing along a complex river network, are proposed. The new algorithms enable to use variable space steps, variable celerity and diffusivity as well as any distribution in space and time of lateral inflow. The accuracy of the new algorithm related to the modified diffusive wave equation is tested by numerical experimentation using nondimensionalized variables and schematic boundary conditions. Three error criteria were chosen to evaluate the numerical methods: maximum discharge, maximum position in time and instability caused by the inflow signal. For the fully diffusive wave flood routing problem, the two algorithms tested (CNX and CNT) gave similar results. Note that the accuracy of a numerical method flood routing depends also on the space and time discretization, the volume and the distribution of lateral inflow.

References