Analytical solution of the depth ratio for forced hydraulic jump

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Abstract

In this paper, an analytical solution for the jump depth ratio via the application of the basic flow equations has been developed, calibrated and verified. The developed equation can be applied in different situations with wide range of applications such as horizontal channel having smooth or rough beds, with or without positive and negative steps. Large sets of experimental data were collected to calibrate and verify the analytically derived model. It is found that the computed results agreed well with the experimental data. It is concluded that the developed equation is easy to apply in addition to its simplicity and accuracy. Also, it can be extended to include the effect of sloping channel floors.

1 Introduction

The hydraulic jump is one of the most frequently encountered phenomenon of the open channel flow. The depth ratio of the hydraulic jump formed on horizontal smooth rectangular channel floor can be computed using the classical depth ratio equation (Belanger equation). Many studies have published concerning the hydraulic jump on horizontal smooth floor. Among these studies, several attempts have made to improve the prediction of Belanger equation by accounting for the boundary roughness, see, e.g. Rajaratnam [16,17], Gill [5] and Hager and Bremen [7]. The effect of rough beds in horizontal channels on the depth ratio of the hydraulic jump is discussed both experimentally and theoretically, e.g., Leutheusser and Schiller [9], Rajaratnam [18], Peterka [15], Abdelsalam et al. [1], Mohamed Ali [10], Negm [11], Negm et al. [12], and Alhamid [2]. Recently, Alhamid et al [3] developed a theoretical equation for computing the depth ratio of the jump formed in roughened sloping channels. Also, some interesting experimental
and theoretical investigations regarding the horizontal stilling basins with positive and negative steps have published, see, e.g. French [4], Hager and Bretz [6], Ohtsu and Yasuda [14], Hager [8] and Negm et al [13]. Although, no information in the literature are available for computing the sequent depth ratio of the forced hydraulic jump formed in horizontal rectangular roughened stilling basins containing positive and negative steps. So, this paper presents an analytically derived equation for this purpose.

2 Model formulation

The derivation of the theoretical equation of the sequent depth ratio of the hydraulic jumps (types A and B) occurred in a horizontal rectangular channel can be developed via the application of the momentum equation at the beginning and at the end sections of the jump (sections 1 & 2 of figure 1). Consider the pressure distribution shown by figure (1a) and assuming uniform velocity distribution and hydrostatic pressure at the two sections, the following equation can be written:

\[ P_2 - P_1 + F_r = \frac{\gamma q}{g} (V_1 - V_2) \]  

(1)

where \( P_1 = 0.5 \gamma y_1^2 \) and \( P_2 = 0.5 \gamma k^2(s + y_2)^2 \) are the hydrostatic pressures at the beginning and at the end of the jump where the water depths are \( y_1 \) and \( y_2 \), \( \gamma \) is the specific weight of water, \( s \) is the step height, \( k_\alpha \) is a coefficient to account for the difference between the actual pressure on the step and the assumed hydrostatic one. The force \( F_r = C \rho A V_1^2 / 2W \) is the resistance force due to the presence of the roughness elements with \( C \) being the drag coefficient, \( \rho \) is the density of water, \( A \) is the area of the channel covered by roughness, \( (A = IL_R W) \), \( I \) is the roughness density, \( (I = aN/L_R W) \), \( V_1 \) is the mean velocity at the beginning of the jump, \( W \) is the width of the channel, \( L_R \) is the length of the channel covered by roughness and \( N \) is the total number of roughness elements. Regarding the A-jump with positive step, equation (1) can be re-expressed as:

\[ \frac{\gamma k^2 (y_2 + s)^2 - \gamma y_1^2}{2} + \frac{\gamma}{2} \frac{V_1^2}{g} CIL_R = \frac{\gamma q}{g} (V_1 - V_2) \]  

(2)

Keeping in mind that the one dimensional continuity equation holds true, i.e., \( (V_2 = V_1 y_1 / y_2) \) and that \( F_i = V_1 \sqrt{g/(gy_1)} \), and expressing in dimensionless form, one can obtain the following equation:
\[
\left( \frac{y_2}{y_1} \right)^2 + \left( \frac{y_2}{y_1} \right) - \left( \frac{2F_1^2}{1+\lambda} - \frac{F_1^2\lambda_r}{1+\lambda} \right) = 0
\]  

(3)

which when solved for \( y_2/y_1 \) yields:

\[
\frac{y_2}{y_1} = \frac{1}{2} \left( -1 + \sqrt{1 + \frac{8F_1^2}{1+\lambda} - \frac{4F_1^2\lambda_r}{1+\lambda} - 1} \right)
\]  

(4)

where \( \lambda \) is termed as the step parameter which accounts for the presence of the step. For \( A \)-jump with positive step, \( \lambda \) is defined by the following expression:

\[
\lambda = \lambda_A^+ = \frac{k_s(S^2 + 2SY)}{Y^2 - 1}
\]  

(5)

in which \( S = s/y_1 \) and \( Y = y_2/y_1 \).

The second term, \( \lambda_r \), is called the roughness parameter which is a function of the length of roughened bed, \( L_r \), the height of the roughness block, \( h_b \), the roughness density, \( I \), and the initial Froude number, \( F_1 \). This parameter is defined by the following expression [11,3]:

\[
\lambda_r = C I \left( \frac{L_r}{y_1} \right) \left( \frac{Y}{Y-1} \right) \quad \text{where} \quad Y = y_2/y_1
\]  

(6)

where \( C \) is a function of \( F_1, L_r, h_b \) and \( I \).

Considering the pressure distribution for the other cases of jumps at positive and negative steps as shown in figures (1b, 1c & 1d), the form of equation (4) will be the same with the following definitions for \( \lambda \):

**For the \( A \)-jump with negative step:**

\[
\lambda = \lambda_A^- = \frac{k_s(S^2 - 2SY)}{Y^2 - 1}
\]  

(7)
For B-jump with positive step:

\[ \lambda = \lambda_B^+ = \frac{k_s S(Y + S)}{Y^2 - 1} \]  

For B-jump with negative step:

\[ \lambda = \lambda_B^- = \frac{k_s S^2}{Y^2 - 1} \]  

In equation (4), it should be noticed that:
(a) the sign of \( \lambda \) is negative for negative jumps and positive for positive jumps.
(b) when \( \lambda_r = \lambda = 0 \), i.e. both the effects of step and roughness are absent, equation (4) becomes the well known Belanger equation which verifies the derived solution.
(c) if the step is absent, \( \lambda \) should be zero and equation (4) becomes suitable for roughened bed without step.
(d) if no roughness exists, equation (4) can be applied for steps on smooth floors, i.e. \( \lambda_r = 0 \).

Calibration and verification of the derived solution

As seen from equations (5,7,8,9), the step parameter, \( \lambda, \) is a function of the pressure profile coefficient, \( k_s, \) sequent depth ratio, \( y_2/y_1, \) and the step height ratio, \( S = s/y_1. \) It is known that both of \( y_2/y_1 \) and \( k_s \) are a function of the initial Froude number and \( S \) [6,14]. Therefore the step parameter can be expressed as a function of \( F_1 \) and \( S \) only in non-dimensional form as follows:

\[ \lambda = f(F_1, S) \]  

Also, from equation (6) and as a matter of fact, the sequent depth ratio, in case of the roughened bed, is a function of \( F_1, L_R, h_b, I. \) Then equation (6) can be re-written in a dimensionless form as:

\[ \lambda_r = f(F_1, \frac{L_R}{y_1}, \frac{h_b}{y_1}, I) \]  

In order to evaluate the parameters given by equations (10) and (11), experimental data were collected from different sources [1,10,6]. Regarding the roughened bed, it was proved that the density of \( I = 10\% \) or about \( 10\% \) is
the optimal to be used in the design of the stilling basins, [1,10,12,2]. Therefore, the data used to evaluate $\lambda_1$ is only for $I=10\%$ and is available in references [1,10] for horizontal roughened beds with $h_b=1.6$ cm and $L_r/h_b$ ranges from 17.81 to 125. Based on these data and using the multiple linear regression analysis the following regression model is obtained.

$$\lambda_r=0.3153 \log \left( F_1^{2.74} \left( \frac{L_r}{h_b} \right)^{-0.105} \right) -1.0013$$

(12)

The computed depth ratios using equation (4) when $\lambda=0$ and $\lambda_r$ as computed from equation (12) are plotted against the observed data in figure (2a) for different $L_r/h_b$. It seems that the values are 5% higher than the actual ones. This deviation is mainly due to the absence of the other roughness factors from equation (12) which affects the depth ratio significantly.

Similarly, equation (10) needs to be evaluated experimentally. The experimental data collected by Hager and Bretz [6] on a negative and positive steps on smooth horizontal channel for the ranges of $3.5<F_1<7$ and $0<S<3.5$ are used for this purpose. For these data the average values of the pressure profile coefficient were as follows: 1.335, 0.942, 2.729 and 2.492 for $A^+$, $A^-$, $B^+$ and $B^-$ jumps. Several attempts were made to fit the $\lambda$ values with a multiple linear regression model with the best coefficient of determination, $R^2$, of about 0.90 and a minimum standard error of estimate, SEE in the range 0.03 to 0.1. The following form is found to fit the data well.

$$\lambda=c_o+c_1F_1+c_2S$$

(13)

where $c_o$, $c_1$ and $c_2$ are constants depending upon the jump type. The values of these constants are as follows: 1.336, -0.369 & 1.099 for $A^+$ jump; 0.370, -0.055 & 0.134 for $A^-$ jump; 1.453, -0.418 & 1.339 for $B^+$ jump; and 0.353, -0.111 & 0.064 for $B^-$ jump.

Figure (2b) presents the computed depth ratios using equation (4) when $\lambda_r=0$ and $\lambda$ as given by equation (13) for the different jump types. This figure shows good agreement between the observed depth ratios [6] and the computed ones using the analytically derived solution of equation (4).

In order to verify equation (4) for both the effects of roughened bed and step, figure (2c) is prepared to present a typical comparison between the present experimental observations and the prediction given by equation (4) and equations (12 & 13) for the following cases: (a) roughened bed alone with $L_r/h_b=28.5$ ($\lambda=0$); (b) $A$-jump and $B$-jump with positive step on smooth bed
(λ_r=0); and (c) A-jump and B-jump with positive step with \( L_R/h_b = 28.5 \) (\( h_b = 2 \text{ cm} \)).

It is clear that satisfactory results are obtained with a maximum absolute error of about 8%. It should be mentioned that the present data are conducted on a laboratory flume of 30 cm wide, 45 cm deep and 12.5 m long with a recirculating system. The discharge was measured by an orifice meter and the water depths were measured by means of precise point gauges (0.1 mm accuracy). The sequent depth is always measured when the water surface becomes level and the length of jump is measured according to Peterka [15]. Ranges of \( 0.5 < S < 2.5 \), \( 4 < F_j < 7 \) and \( L_R/h_b = 28.5 \) were covered. For more details on jumps definitions and formations, references [6] and [13] can be consulted.

Conclusions

A mathematical model is developed to predict the sequent depth ratio of the hydraulic jumps formed on horizontal roughened floor with or without steps, equation (4). The model contains two experimental parameters, one to account for the effect of the roughness and the second to present the effects of the steps both positive and negative steps. Based on the available experimental data, statistically derived equations for these parameters are presented (eqs. 12 and 13). The prediction of these equations is valid for the ranges of \( 3.5 < F_i < 7 \), \( 0 < S < 3.5 \) and \( 17.81 < L_R/h_b < 125 \). However, equation (4) can be used for any range of \( F_j \) and \( S \) provided a proper estimation of the parameters \( \lambda \) and \( \lambda_r \) is made for the desired range. It should be noted that equation (4) can be used to compute the depth ratio of a sloping roughened channels with or without steps if a third parameter to account for the effect of slope is included and estimated properly.

References

3. Alhamid, A.A., Husain, D. and Negm, A.M. Prediction of the sequent depth ratio of hydraulic jump on rough sloping channel...


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Figure 1. Definition sketch for different types of jumps (a) A-jump with positive step (b) A-jump with negative step (c) B-jump with positive step and (d) B-jump with negative step

Figure 2. Experimental data versus (a) Eq.(4) with $\lambda$, of eq (12) and $\lambda=0$, (b) Eq.(4) with $\lambda$ of eq (13) and $\lambda=0$, and (c) Eq.(4) for both effect of $\lambda$, and $\lambda$. 