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Numerical simulation reliability; quality assurance in finite element analysis - a review

L. Amati[a], A. Di Anselmo[a], C. Margheriti[a], J.M. Kenny[b]

[a]*ISRIM - Instituto Superiore di Ricerca e Formazione sui Materiali Speciali per le Tecnologie Avanzate, Loc. Pentima Bassa 21, 05100 Terni, Italy*

[b]*Dipartamento di Ingegneria dei Materiali e della Produzione, Università di Napoli, Piazzale Tecchio, Napoli, Italy*

ABSTRACT

Given the ever incresasing interaction between numerical simulation and experimental measurements; it is of vital importance to ensure numerical results to be reliable. Having commercial codes reached a high level of user friendliness, an essential issue is now quality assurance. The final goal of a quality assurance process in fea is to reduce the error, implicit in any numerical procedure, below a pre-determined value in order to validate the accuracy of the results. The present work is focused on the a-posteriori error analysis. As a matter of fact a-priori measures alone are not enough to tell the fea user about the size of the error contained in the analysis while benchmark tests can only ensure a global validation of a finite element package. Reviewing what has been done in the last years to improve the accuracy of a finite element solution, this paper tries to point out the principal results achieved so far and indicate future developments.

1 - INTRODUCTION - Quality Assurance (QA) In Finite Element Analysis

The final goal of a QA process in FEA is to reduce the error made in evaluating a prescribed quantity (i.e., in structural analysis, displacements or stresses) below a pre-specified value in order to ensure a safety coefficient factor to be guaranteed as required in any engineering type of analysis. Besides, the QA process is essential to make the FE user aware of the approximations contained in a numerical method.

Working on quality assurance in FEA there are three main directions to explore:

1) Benchmark tests
2) A-Priori error analysis
3) A-Posteriori error analysis - Adaptivity

1) - Benchmark tests [Dav89, MNH85] have mainly the purpose of:

a) testing various part of a Finite Element system in order to ensure a global validation (Software Quality)

b) educate the user to be aware of the possible limitations of the system

providing, at the same time, an occasion to gain some insights on how the system works

2) - A - Priori error estimation can be carried out without any knowledge on the specific analysis thus giving general and economical pieces of information. It can be divided in three main parts:

a) Quality of Finite Element modelling [MN86, MN87], i.e. accurate representation of the real structure and the loads which has to carry. A poor description of the geometry and / or the loads can affect the order of approximation of a FE solution invalidating all the efforts of the error analysis to produce more accurate results. It is expected that expert systems will help to solve this problem and some work in this direction is being carried out especially from the automotive and aerospace industries. Expert systems are generally based on a large number of FE analysis and on designers' experience and provide a series of " check list " each one suitable for a particular kind of structure (e.g. car frame, helicopter etc.)

b) Control on element distortion parameters like skew, aspect ratio and warping [Bar89, UH89, Rob87, Rob88]. Different types of load conditions can also affect the quality of the analysis i.e. an element could be suitable to represent the stress field arising from one load condition but too large to model correctly another load situation, so that when compiling a list of the requirements an element should satisfy, all different type of loadings must be taken into account.

c) The A - Priori error estimation as intended in the classical sense, i.e. the prediction of what kind of error will be made when analyzing a physical problem, whose mathematical representation is given by a linear partial differential equation, with a numerical technique . Most of the work in this direction has been carried out by mathematicians [BKS79, BS80, Sza90] and some practical applications are now coming out , particularly when this a-priori information are used in combination with the hierarchical type of finite element formulation [ZW89, ZW90, Sza90, ODRW89].

3) A - Posteriori Error Analysis - Adaptivity. The a - posteriori error analysis and successive mesh refinement are the last step of the quality assurance process. To be able to predict the error accurately is clearly essential. A - priori measures alone are not sufficient to tell the user what kind of error is contained in his analysis especially when dealing with complex structures where for example the stress distribution can only be roughly predicted and some elements in the mesh can be too large to capture the stress variation. Unfortunately at the present stage many questions have still been left unanswered. Only to mention few of them:

What kind of quantity is to be considered when estimating the error ? This is a very delicate point. In fact the user might be interested in having a displacement or a stress field correctly described ; however looking only at say the stresses, it would be be possible to neglect zones of a structure that have been poorly modelled thus " locking " the energy in large elements [Pea89]. In this case a finer mesh would lead to an energy release and possibly to a redistribution of displacement and stress fields.

What kind of finite element formulation is more suitable for error estimation? at present h, p and h-p combination are being assessed and results seem to show that there is no optimal version [Sza90].

Finally, will mathematicians and engineers try to join their efforts? The different approach, different notation and jargon used to deal with the same problem is striking. Some attempts have been made so far (see e.g. [GKZB83

, KGZB83 , ZC86, Bab90]) but for most engineers bilinear forms, norms and Hilbert spaces are esoteric names (the vicecersa could be true perhaps about concepts like strain energy or compatibility) and an attempt to try to use a common vocabulary would be extremely helpful.

2 - Error Analysis for the Finite Element Method - an Overview

Error estimation in FEA has received a great deal of attention especially in the last ten years and trying to summarize the results achieved so far is not an easy task. During our research we found extremely helpful the work of Gago [Gag82] which aided as a guide trough the development of error analysis during its early stage in the seventies. The proceedings of the workshop on reliability on computational mechanics held in 1989 at the University of Texas at Austin [CMAME90 vol82] together with the book on accuracy estimates and adaptive refinement in finite element computation [AEARFEC86] contain many pieces of information about the research carried out in the last decade.

A Brief History

During the first period when some research to improve the accuracy of a FE solution was being done in the early seventies the problem tackled was to find an optimal mesh for a given number of elements. According to Gago [Gag82], Mc Neice and Marcal [MNM71] were the first to study this kind of problem. Their approach consisted in writing the total potential energy :

$$\Phi = \frac{1}{2}\left[\mathbf{p}^T\left(\int_V \mathbf{B}^T\mathbf{D}\mathbf{B}\, dV\right)\mathbf{p}\right] - \mathbf{q}^T\mathbf{p} \qquad (2.1)$$

regarded as a function of both nodal displacement $\mathbf{p}$ and nodal coordinates $\mathbf{c}$. Imposing the stationary conditions :

$$\frac{\partial \Phi}{\partial \mathbf{p}} = 0 \qquad (2.2)$$

$$\frac{\partial \Phi}{\partial \mathbf{c}} = 0 \qquad (2.3)$$

the first set of equations leads to the standard FE system of equations while the second characterises the optimal node point location problem.

However the global system of equations becomes non-linear and the amount of computational effort grows as the third or the fourth power of the number of DOFs of the model thus rendering useless the estimation of a better grid distribution instead of running a second analysis.

Oliveira [Oli71] proposed an alternative algorithm where the non-linear minimization process could be performed iteratively. Oliveira suggested that the optimal mesh should be based on constant strain contours thus indicating a criterion to position the nodes along the isoenergetics (i.e. constant strain)

lines. A similar approach has recently been rediscussed by Wiberg and Zeng [WZ89] in conjunction with the advancing front adaptive technique. The basic idea implies the recovery of strain (or stress) gradient and the use of a-priori error estimate to refine the mesh.

The first attempt to introduce new DOFs in the refined mesh is found in the papers by Melosh and Killian [MK76] and Peano et al [Pea78]. Melosh and Killian try to estimate the difference in energy between the current and the next solution while Peano recovers an energy gradient. The two approaches are quite similar and have both been classified as " forward energy difference methods " .

The strategy of Melosh and Killian is based on an a-priori assumption of the behaviour of the energy error norm given by the formula :

$$|| e ||_E^2 = Ae^{BD} \tag{2.4}$$

$|| e ||_E^2$ is the energy error norm D is the no of DOF and A and B are two constants to be determined from two successive solutions. However there is no guarantee for the error to behave exponentially so that the estimate is not necessarily accurate. A similar strategy has been adopted more recently in the error estimator proposed by Szabo [Sza86, Sza90]

Peano's error estimator instead is based on the particular structure of the stiffness matrix and the load vector when using the p-version of FEM [PR78]. The basic idea is to estimate an energy gradient for each new DOF introduced and refine the mesh where this gradient is higher (see next paragraph).

The main drawback of this error estimation procedure is the necessity of computing all additional stiffness terms. The inconvenient could be removed by considering a " backward energy difference " where an energy gradient is found considering the current energy and a " restrained energy " based on a subset of displacements, however this method is not very accurate [PR78]. Nevertheless this error estimation technique has the advantage of requiring little computational effort and as it has already been mentioned in chapter 1, working on energy basis it is possible to spot overconstrained zone artificially created by poor modelling which are able to affect the stress distribution in the structure [Pea89]. Recently [DLY88], the same type of estimator has also been analysed from a mathematical point of view and adopted for 2 - D linear elliptic problem in conjunction with an iterative solution technique.

An alternative formula always using the p- version to recover the energy gradient avoiding the computation of additional stiffness terms was suggested by Zienkiewicz et al [ZGK83] together with an error estimator presented as a norm (see paragraph 2.3).

A further step in the error analysis was introduced when researchers started to look at the current solution and the residual **r** at the equilibrium equation level :

$$\mathbf{L}^T\sigma + \mathbf{f} = \mathbf{r} \tag{2.5}$$

where **L** is a linear operator **s** are the stresses deriving from the FE solution and **f** are known body forces / unit volume. The attention then focused on how to evaluate the residual and what kind of measure to estimate the error. Babuska and Rheinboldt [BR78] gave, for the first time, a mathematical basis to error analysis. The error is computed on a local basis and is based on two terms one

related to the stress jump at the boundaries of the element and the other to the internal residual in each element :

$$\varepsilon_e^2 = c_{e1}\int_\Omega r^2 d\Omega + c_{e2}\int_\Gamma J^2 d\Gamma \tag{2.6}$$

where the c_{e1} and c_{e2} are a function of the problem type and the type of element and r and J the domain and boundary residual respectively. With

$$\varepsilon^2 = \sum \varepsilon_e^2 \tag{2.7}$$

as the total error estimate, it is then possible to show that when the solution has entered the asymptotic range :

$$C_1 \varepsilon \leq \| e \|_E \leq C_2 \varepsilon \tag{2.8}$$

where the C's are constants independent of the mesh and the particular solution and $\| e \|_E$ the error in energy norm i.e., in the specific case of elasticity the square root of the error in strain energy :

$$\| e \|_E = \left(\int_\Omega (\mathbf{Le})^T \mathbf{D} (\mathbf{Le}) \, d\Omega \right)^{1/2} \tag{2.8a}$$

where $\mathbf{L}$ is a linear operator, $\mathbf{u}$ are the exact displacements,

$\hat{\mathbf{u}}$ the displacements obtained with the finite element analysis,

$\mathbf{e} = \mathbf{u} - \hat{\mathbf{u}}$ and $\mathbf{D}$ is the modular matrix.

A complete mathematical proof has been given only for linear elements in 1-D and 2-D and when the solution has entered the asymptotic range.

Kelly [Kel84] has adopted a strategy based both on the residual and on the complementary energy to obtain an upper bound for the error measured in an energy norm. The residuals in the domain and on the boundary are regarded as forcing functions in the governing differential equation and boundary conditions. The FE solution is then the exact solution to a perturbed problem with a supplementary pseudo - force system. This system is globally self equilibrating as indeed is the FE solution. As an example, an elliptic boundary value problem can be considered [KMRM87] ; if u and u_h are the exact and the approximated solution and n is the normal direction to the boundary the problem is formulated as:

$$\nabla^2 u = -f \quad \text{on the domain } \Omega \tag{2.9}$$

$$\frac{\partial u}{\partial n} = g \quad \text{on } \Gamma_g \text{ (portion of the boundary with prescribed forces)} \tag{2.10}$$

$$u = 0 \text{ on } \Gamma_\phi \text{ (portion of the boundary with prescribed displs)} \tag{2.11}$$

Defining the error e as $e = (u - u_h)$ and being able to partition the self equilibrating system at an element level, it is possible to determine the local error from:

$$\nabla^2 e = -r \quad \text{on the domain } \Omega_i \tag{2.12}$$

$$\frac{\partial e}{\partial n} = g - \frac{\partial u_h}{\partial n} \quad \text{on } \Gamma_{gi} \tag{2.13}$$

$$e = 0 \text{ on } \Gamma_{ui} \tag{2.14}$$

The complentary energy principle applied to the error is then used to ensure that the estimate of the energy of the error is an upper bound.

The main difficulty arises on how to determine the partitioning (splitting factor) for the term $\partial e / \partial n$ between two adjacent elements. In his work Kelly shows that this can be done for one dimensional linear elements while difficulties of insuring the splitting factor to behave well are encountered for two dimensional bilinear elements. Recently Kelly and Isles [KI89] have also indicated an alternative way of estimating the local error extrapolating the solution to the energy bound instead of using the complementary energy.

Another original approach that has been proposed is the one by Lavedeze et al [LCP86]. The method of estimating the error is based on the assumption that while boundary conditions and equilibrium equations can be considered as exact when modelling a structure, the lack of accuracy in defining the constitutive law is a major unknown. Starting from the results obtained from the FE solution a consistent stress and displacement field (i.e. such as to satisfy the kinematic constraint and the equilibrium equation respectively) is recovered. The quality of the FEA is then measured as the lack of satisfaction of the constitutive relationship showed by this solution.

It was probably the work on finding an optimal way to recover a consistent stress distribution [see e.g. OB71] added to discussions on local and global smoothing [HC74] and on iterative procedure [ZXN85,Cri86 p.27] to reduce the computational effort to compute this improved stress field that led Zienkiewicz and Zhu [ZZ87] to elaborate a new type of error estimator based on a smoothing projection technique to recover a more accurate stress distribution.

The main point is the recovery of a better approximation of the stress field obtained from the current solution via the interpolation:

$$\sigma^* = \mathbf{H}\, \bar{\sigma}^* \tag{2.15}$$

where $\mathbf{H}$ is the shape function matrix and σ^* is the smoothed stress

field. In order to compute the parameters $\bar{\sigma}^*$ an equality between the computed and the smoothed stresses can be imposed in a mean sense:

$$\int_\Omega \mathbf{H}^T (\hat{\sigma} - \sigma^*) \, d\Omega = 0 \qquad (2.16)$$

where $\hat{\sigma}$ are the FE stresses. It can be noticed that the projection given by (2.16) is equivalent to the least squares fit or minimization of

$\Phi = \int_\Omega (\hat{\sigma} - \sigma^*)^2 \, d\Omega$ with respect to $\bar{\sigma}^*$ [ZT89 p. 347, HC74]

Substituting (2.15) in (2.16) :

$$\bar{\sigma}^* = \mathbf{A}^{-1} \int_\Omega \mathbf{H}^T \hat{\sigma} \, d\Omega = \mathbf{A}^{-1} \int_\Omega \mathbf{H}^T \mathbf{D} \mathbf{L} \mathbf{H} \, d\Omega \, \mathbf{p} \qquad (2.17)$$

where $\mathbf{A} = \int_\Omega \mathbf{H}^T \mathbf{H} \, d\Omega$

To save the computational effort of inverting the matrix **A** a lumped form of this matrix can be used or solution of (2.17) can be carried out iteratively [ZXN85]. It can be noticed at this point that either local smoothing plus nodal averaging [HC74] or even a cruder nodal averaging with no global solution are often used for stress recovering. Once the smoothed stress field has been recovered the error in the energy norm can be computed as :

$$\| e \|_E = \left(\int_\Omega (\mathbf{e}_\sigma^T) \, \mathbf{D}^{-1} \, (\mathbf{e}_\sigma) \, d\Omega \right)^{1/2} \qquad (2.18)$$

where $\mathbf{e}_\sigma = \sigma^* - \hat{\sigma}$

The method is sufficiently simple and it has been succesfully applied to bidimensional problems together with an adaptive algorithm for triangular linear elements and more recently to plate bending [ZZ89,1] and shell [YPW90] element types.

A proof of the equivalence of this estimator to the one devised from Babuska in case of bilinear elements has been given [ZR87] , recently a more mathematical treatment of the asymptotic convergence of the estimator has also been discussed [AZCZ89].

If the Zienkiewicz - Zhu error estimator is based on the difference in stresses between adjacent elements the one proposed by Bathe and Chae [BC89]

focuses on the residual inside the element. Bathe and Chae have found that in case of plane stress problems in presence of stress singularities using higher order elements the contribution to the error estimation given by the internal residual is much larger than the one given by the stress jumps term. In concept an error estimator based on the internal residual is quite simple but difficulties in reaching a method that shows both high efficiency and reliability for general applications have been envisaged by the authors themselves.

Recently Zienkiewicz and Zhu have presented another error estimator based on stress recovery [ZZ89,2]. The technique used for this estimator originates from the superconvergence properties of the FE solution in some points in the interior of the element which are known a-priori. This error estimator is based on local information and has been succesfully tested also on 1-D higher order elements. Doubts exist for its succesful use in more than 1-D because the superconvergence properties may be lost [Lev85].

In the early eighties some papers have begun to investigate on h and p - version convergence rate when increasing the number of DOFs. The reports by Babuska and Szabo [BKS79, BS80, Sza86] had led to the conclusion that in problems where the singularities are not inside the domain of the FEs the rate of convergence of the p - version is twice that of the h - version. More recently Szabo has made a further investigation and has divided structural problems in three categories according to the type of singularity [Sza90] .

In the first category the exact solution is analytic (i.e. can be expanded from a point into a Taylor series about that point) on each FE including its boundaries. For problems which fall in the second category, the exact solution is analytic on each finite element including its boundaries with the exception of some of the vertices which are called singular points; for problems strongly in this category the maximum stress is infinity at a singular point, for problems weakly in this category the stress is finite over the entire domain. To the third category belong the problems for which a mesh cannot be constructed so that singular points are only on the vertices or the locations where abrupt changes occur in the derivatives of the exact solution, such as material changes, are at interelement boundaries.

For problems in the first category the most effective method for controlling the errors of approximation is given by the p-version of FEM since the error in the energy norm behaves exponentially :

$$\| u_{Ex} - u_{FE} \|_E \leq \frac{k}{\exp\left(\gamma N^{\theta} \right)} \tag{2.19}$$

where k, g and q are positive numbers and N is the number of DOFs. H - version of FEM can only achieve an algebraic rate of convergence :

$$\| u_{Ex} - u_{FE} \|_E \leq \frac{k}{N^{\min(p,\lambda)/2}} \tag{2.20}$$

where again k is positive and so is the exponent of N, being p the polynomial degree of the shape functions used and l a fractional number depending on the type of singularities in the problem examined. For problems belonging to the second category, the most effective method is the combined h - p version using geometric meshes i.e. meshes graded towards the singular point

with geometric progression. H - version can achieve the faster rate of convergence for problem strongly in the third category while for problem weakly in third category p or h p -version can be used.

Given the fact that h - version performs better for a certain class of problems and in general when the asymptotic range of the solution has not been entered (typically when the error in a stress norm is of about 5 - 10 % [ZZG89]) whereas the p - version is faster in problems where limited or small singularities are present, some recent research efforts have been devoted to the analysis of a combined h - p version. In 1987 Rank and Babuska [RB87] have given an indication of the ingredients which should be contained in an h - p based expert system. More recently some adaptive procedures have been proposed by Wiberg and Zeng [WZ89] and Zienkiewicz et al [ZZG89] on 2-D elastic type of problems. An extensive review of error estimator for the h-p version has been published by Oden et al [ODRW89].

It appears obvious that one of the main difficulties in using this kind of approach is given by their implementation in a commercial code but as it has been said an optimal rate of convergence can be achieved.

Error estimation for the p - version of the finite element method

Most of the error estimators mentioned in the previous paragraph have been devised for the h - version of the FEM, mainly due to the fact that the vast majority of commercial codes available at present are based on this type of formulation.

However due to some characteristics like the better conditioning of the stiffness matrix, the rate of convergence to the exact solution that can be achieved and the need to only define the mesh once (it is worth mentioning that at present the mesh modelling takes about 45% of the total time in a finite element process [Dav89]) has raised new interest in the p-version of the FEM. The next paragraph illustrates in some detail, following a chronological order, some of the error estimators that have so far been presented for this method.

Peano et al [PR78] have been the first to investigate the idea of estimating the error using the hierarchical structure of the stiffness matrix. If the total potential energy is written as it derives from the FE formulation one obtains :

$$\Phi (\mathbf{p}) = \frac{1}{2} \mathbf{p}^T \mathbf{K} \mathbf{p} - \mathbf{p}^T \mathbf{q} \qquad (2.21)$$

Using the hierarchical formulation and denoting with the subscrits o and n the old and the new DOFs respectively

$$\mathbf{K} \begin{bmatrix} \mathbf{K}_{oo} & \mathbf{K}_{on} \\ \mathbf{K}_{no} & \mathbf{K}_{nn} \end{bmatrix} \mathbf{p} \begin{bmatrix} \mathbf{p}_o \\ \mathbf{p}_n \end{bmatrix} \mathbf{q} \begin{bmatrix} \mathbf{q}_o \\ \mathbf{q}_n \end{bmatrix} \qquad (2.22)$$

The energy gradient component in the direction of the single variable p_i

is :

$$\frac{\partial \Phi}{\partial p_j} = \sum_{k=1}^{i} k_{jk} p_k - q_j = r_j \qquad (2.23)$$

If p_i is one of the additional DOFs and the current solution is considered (i.e. $\{ \mathbf{p}_o = \bar{\mathbf{p}}_o \}$ and $\{ \mathbf{p}_n = \mathbf{0} \}$) the gradient is computed from the current solution and can be regarded as the residual of the equilibrium equation for the j - th variable. Peano then introduces a scaling factor to normalise the shape functions to unit strain energy

$$R_j = \frac{r_j}{\sqrt{k_{jj}}} \qquad (2.24)$$

if R_j is less than a prescribed tolerance R_1 the additional variable p_i is discarded otherwise a new DOF is introduced.

Recently, Deuflhard, Leinen and Yserentant [DLY88], have given a mathematical proof of the convergence property of this error estimator for 2 - D elliptic type of problems analysed with triangular elements.

In 1983 Zienkiewicz, Gago and Kelly [ZGK83, ZC86, ZT89 pp426] have noticed that the residual r_j can also be expressed as

$$r_j = \int_\Omega h_j \, r \, d\Omega \qquad (2.25)$$

where h_j is the shape function related to the j - th variable and r is the residual obtained by substituting the FE approximation in the equilibrium equation. This expression for the residual leads to an error estimator in the energy norm for the single j - th new DOF considered given by

$$\| e \|_E = \left(\int_\Omega (\mathbf{Le})^T \mathbf{D} (\mathbf{Le}) \, d\Omega \right)^{1/2} = \frac{\int_\Omega h_{i+1}^2 \, d\Omega \int_\Omega r^2 \, d\Omega}{k_{jj}} \qquad (2.26)$$

Using this formula is not necessary to compute the new stiffness terms unless required for the next analysis, however difficulties arise when this expression is computed at the boundaries of each single element. The integral (2.25) must be replaced by the sum

$$r_j = \int_{\Omega} h_j r \, d\Omega = \int_{\Omega_{int}} h_j r \, d\Omega_{int} + \int_{\Gamma} h_j J \, d\Gamma \qquad (2.27)$$

where Ω_{int} and Γ represent the inside and the boundary of the FE domain respectively

The integral at the boundary suffers from the same inconvenient mentioned for the complementary energy approach i.e. the splitting factor of the value of this integral is not known. Clearly the simplest way is to compute the integral only once and divide the contribution equally between adjacent elements.

The error estimation technique proposed by Szabo [Sza86, Sza90, ZZG89] is based on the a - priori estimates of the convergence of the hierarchical formulation of the FEM associated with a uniform p-refinement.

Considering the square of the error of the energy norm it can be shown that :

$$\| e \|^2_E = \Phi_p - \Phi \qquad (2.28)$$

where Φ_p is the potential energy evaluated for the computed solution at the polynomial level p and Φ is the potential energy for the exact solution. Using the p- version of the FEM for problems with singularities lying on the boundary of the FE domain an algebraic rate of convergence for the error in the energy norm can be asimptotically achieved

$$\Phi_p - \Phi \leq \frac{k^2}{N_p^{2\beta}} \qquad (2.29)$$

where N_p is the number of DOFs at the polynomial level p and k and b are positive constant. Using (2.29) on three successive FE solutions it is possible to eliminate the two constants k and b obtaining the relationship

$$\frac{\Phi - \Phi_p}{\Phi - \Phi_{p-1}} \approx \left(\frac{\Phi - \Phi_{p-1}}{\Phi - \Phi_{p-2}} \right)^Q \qquad (2.30)$$

where Q only depends on N_{p-2}, N_{p-1} and N_p

$$Q = \frac{\log (N_{p-1} / N_p)}{\log (N_{p-2} / N_{p-1})} \qquad (2.31)$$

To estimate the exact potential energy F the non - linear equation (2.30) has to be solved.

The estimator has been applied also to problems without singularitues and outside the asymptotic range giving accurate estimates for the total energy and consequently for the error in the energy norm [Sza90, ZZG89]. It has to be noticed however that this estimator only gives a global measure of the error so that it is not possible to refine the mesh adaptively.

Remarks

As a summary on what has been said in this chapter and to give to the error analysis the right " order of magnitude " in the global process of designing a structure it is helpful to mention a paper recently published by Babuska [Bab90]. In that paper the following flow chart illustrates the design process :

1 PHISYCAL PROBLEM AND CRITERIA
2 BASIC MATHEMATICAL PROBLEM
3 SIMPLIFIED MATHEMATICAL PROBLEM AND ANALYSIS OF THE ERRORS CAUSED BY THE SIMPLIFICATION
4 NUMERICAL TREATMENT AND ANALYSIS OF THE ERRORS CAUSED BY NUMERICAL TREATMENT
5 PHISYCAL CONCLUSIONS AND ENGINEERING DECISION

Dealing with FEA only includes step number 4 and if the designer is not aware of how accurate the basic mathematical problem is in representing step number 1 and what order of approximation is introduced in step number 3, the analysis of the error involved in the FEA could lead to the wrong conclusions and decisions in step number 5. The engineering knowledge and understanding will not be substituted [ZZ90] with any expert system and will always be the main ingredient for a safe design, but it is clear that the use of hierarchical model [BLB90] or similar concept able to follow the path illustrated above will play a key role in helping the designer through a complete understanding of the whole process.

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Symbols

AEARFEC Accuracy estimates and adaptive refinement in finite element computations
CANM Communications in Applied Numerical Methods
CMAME Computer Methods in Applied Mechanics and Engineering
IJNME International Journal for Numerical Methods in Engineering

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